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Winterbottom Construction for Finite Range Ferromagnetic Models: An $\mathbb{L}_1$-Approach

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We provide a rigorous microscopic derivation of the thermodynamic description of equilibrium crystal shapes in the presence of a substrate, first studied by Winterbottom. We consider finite range ferromagnetic Ising models with pair interactions in dimensions greater or equal to 3, and model the substrate by a finite-range boundary magnetic field acting on the spins close to the bottom wall of the box.

KEY WORDS: Ferromagnetic lattice models; Equilibrium crystal shapes; Wulff construction; Boundary Gibbs fields; Winterbottom construction.

1. INTRODUCTION AND RESULTS

1.1. Winterbottom Variational Problem

The Winterbottom theory[(1)] gives a phenomenological prediction for the equilibrated shapes of small (that is disregarding gravitation) crystal particles placed on solid substrates. The notion of equilibrium comes on two levels:

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(i) As far as the bulk properties are considered, both the particle and the vapor around it are assumed to represent two different *thermodynamic phases* of the same physical substance.

(ii) The total interfacial free energy of the system comprises the anisotropic *surface tension* between the particle and the vapor, as well as the *boundary surface tensions* between the particle and the substrate and, respectively, between the vapor and the substrate. The substrate could, thus, exhibit a preference towards one of the phases, and the macroscopic Winterbottom equilibrium shape corresponds to the state of minimal interfacial energy at a given particle volume.

We use β to denote the inverse temperature at which the two phases (crystal and vapor) coexist, and the parameter η to quantify the preference of the substrate towards one of the two pure phases. In the magnetic interpretation of the microscopic models we shall consider below η is the boundary magnetic field. According to the suppositions (i), (ii) above, the two principal players to determine the equilibrium shape are the anisotropic crystal-vapor surface tension $\tau_\beta = \tau_{\beta,\mathrm{CV}}: \mathbb{S}^{d-1} \mapsto \mathbb{R}_+$ and the number $\Delta_{\beta,\eta} = \tau_{\beta,\mathrm{SC}} - \tau_{\beta,\mathrm{SV}}$, where $\tau_{\beta,\mathrm{SC}}$ and $\tau_{\beta,\mathrm{SV}}$ are, respectively, the substrate-crystal and the substrate-vapor boundary surface tensions. Indeed, if the crystal particle occupies the region P of $\mathbb{R}^d$ its boundary splits into two disjoint pieces $\partial P = \partial P_{\mathrm{SC}} \cup \partial P_{\mathrm{CV}}$ (Fig. 1), and, up to a constant, the total surface energy of the system can be written (at least for piece-wise smooth shapes) as

$$\mathscr{W}_{\beta,\eta}(P) \triangleq \int_{\partial P_{\mathrm{CV}}} \tau_\beta(\vec{n}_x)\,\mathrm{d}S(x) + \Delta_{\beta,\eta}\,|\partial P_{\mathrm{SC}}|. \tag{1.1}$$

The equilibrium particle shapes at a prescribed volume v correspond, in the above notation, to the minimizers of the following variational problem:

$$\mathbf{(WP)}_{\beta,\eta} \quad \mathscr{W}_{\beta,\eta}(P) \longrightarrow \min \quad \text{Given} \quad \mathrm{vol}_d(P) = v.$$

Winterbottom[1] described the geometric construction of the solutions to $\mathbf{(WP)}_{\beta,\eta}$, which, in fact, is an easy consequence of the general Wulff construction[2] of the equilibrium crystal shapes: First of all the minimizers are scale invariant under dilatation. In order to construct the unnormalized Winterbottom shape pick the "free" Wulff crystal

$$\mathscr{K}_\beta \triangleq \bigcap_{n \in \mathbb{S}^{d-1}} \{x \in \mathbb{R}^d \mid (x, n) \leqslant \tau_\beta(n)\},$$

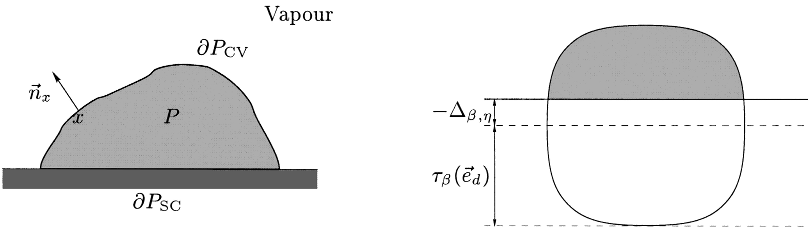

Fig. 1. Left: The crystal P on the substrate. The boundary is decomposed into two pieces, corresponding to the crystal/vapor interface and the crystal/substrate interface. Right: The Winterbottom shape $\mathscr{K}_{\beta,\eta}$ (shaded region) is obtained by intersecting the Wulff shape and the half-plane; on the picture $\Delta_{\beta,\eta}$ is negative.

and intersect it with the half-space

$$\mathscr{K}_{\beta,\eta} \triangleq \mathscr{K}_\beta \cap \{x \in \mathbb{R}^d \mid x_d \geqslant -\Delta_{\beta,\eta}\}, \tag{1.2}$$

where we use the coordinate representation $x=(x_1,..., x_d)$ of a vector $x \in \mathbb{R}^d$. It is convenient to parameterize the substrate surface by one of the coordinate hyperplanes, say by $\mathscr{P}_d \triangleq \{x : x_d = 0\}$. The equilibrium Winterbottom shape at volume v is, then, given by

$$\mathscr{K}_{\beta,\eta}(v) \triangleq \left(\frac{v}{\lambda}\right)^{1/d} (\Delta_{\beta,\eta}\vec{e}_d + \mathscr{K}_{\beta,\eta}), \tag{1.3}$$

where $\vec{e}_d$ is the unit coordinate vector in the d-direction and $\lambda = \lambda(\beta,\eta) = \mathrm{vol}_d(\mathscr{K}_{\beta,\eta})$ is used to denote the volume of the unnormalized Winterbottom shape.

As we shall explain in Subsection 1.5, the value of the difference $\Delta_{\beta,\eta}$ always lies in the interval $[-\tau_\beta(\vec{e}_d), \tau_\beta(\vec{e}_d)]$.[4] Thus, one of the following three cases happens:

1. *Complete drying:* $\Delta_{\beta,\eta} = \tau_\beta(\vec{e}_d)$. In this case $\mathscr{K}_{\beta,\eta} = \mathscr{K}_\beta$, which means that the Winterbottom droplet has no energetic reasons to be near the substrate surface and can be located anywhere above it.

2. *Partial drying/wetting:* In this regime $\Delta_{\beta,\eta} \in (-\tau_\beta(\vec{e}_d), \tau_\beta(\vec{e}_d))$. Accordingly, the convex body $\mathscr{K}_{\beta,\eta}$ is obtained by removing a proper cap from $\mathscr{K}_\beta$ (Fig. 1). We shall, furthermore, distinguish between the partial drying, $\Delta_{\beta,\eta} > 0$, and the partial wetting, $\Delta_{\beta,\eta} \leqslant 0$, cases.

3. *Complete wetting:* $\Delta_{\beta,\eta} = -\tau_\beta(\vec{e}_d)$. In this case, $\mathscr{K}_{\beta,\eta}$ has volume zero and, therefore, cannot be rescaled to get a volume v. Physically this

[4] We shall always assume that $\tau_\beta(x) = \tau_\beta(-x)$.

corresponds to the formation of a microscopic film along the substrate, preventing the vapor phase from reaching it.

Our main objective here is to give a rigorous derivation of the phenomenological Winterbottom picture in the scaling limit of microscopic models of lattice gases, that is directly from the basic principles of the Equilibrium Statistical Mechanics. The class of microscopic models we consider are ferromagnetic Ising type models with finite range of interactions. In the subsequent subsections we shall briefly recall these models as well as the thermodynamic procedures leading to the notion of *bulk phases* and various *surface tensions* which appear in the functional (1.1). The results fall in the framework of the $\mathbb{L}_1$-theory of phase segregation, which has been initiated in the works[3] on the nearest neighbor Ising model and in the works[4–7] on the Kac Ising models. Recent developments of the theory have been prompted by the work[8] and comprise a rigorous derivation of the (free) multi-dimensional Wulff crystal shapes in the context of Bernoulli bond percolation[8] and in the context of nearest neighbor Ising model in refs. 9 and 10. Most recently the theory has been generalized to the case of symmetric q-states Potts models.[11] We refer to the review article[12] for a detailed account of the $\mathbb{L}_1$-theory and its relation to the Dobrushin–Kotecký–Shlosman approach (see e.g., refs. 13–18) to the problems of phase segregation.

1.2. Other Microscopic Derivations of the Winterbottom Construction

The derivation of the macroscopic geometry of an equilibrium droplet in contact with a substrate has been accomplished in several models, which can be divided into two classes.

The first class is that of SOS-type models, with a positivity constraint, a fixed volume under field, and an attraction to the wall. The first such study was done for the $1+1$-dimensional model in ref. 19 (see also ref. 20 for an alternative approach), while its higher dimensional counterpart was treated in ref. 21; see also ref. 22 for an analysis of the effect of substrate roughness in $1+1$ dimensions. Other related works, which neglect the interaction between the substrate and the interface, are refs. 23–25; notice however that these works cover only one special point of the full regime of complete wetting. This class of models provide a somehow appealing description of this phenomenon, but has several shortcomings. The first one is that the full phenomenology of the wetting transition cannot be observed, due to the very particular constraint that the droplet has to be the graph of a function (e.g., it is of course impossible for the droplet to detach from the wall). The corresponding variational problem also displays some non-physical properties, e.g., the minimizer are in general not scale invariant. And of course, these SOS models are effective models for interfaces, not truly microscopic ones, so they do not clarify the mechanism which generates the interface itself. On the other hand, due to these simplifying features, it is possible to obtain some additional information about fluctuations around the equilibrium shape, see ref. 24, which go beyond what we can achieve for the models considered in the present work. For an in-depth review and references on these effective models, we refer the reader to the survey.[26]

The second class contains more realistic lattice gas models of the type we consider here. Only the two-dimensional nearest-neighbor Ising model had been studied before,[27, 17] but the description provided is rather complete. The wetting transition can be fully described, and the characterization of typical configurations is very accurate, especially when these results are combined with the local limit estimates of ref. 18: There is a unique macroscopic droplet whose boundary is close in Hausdorff distance to one of the solutions of the Winterbottom variational problem, and all the other droplets are of size at most $\log L$, L being the linear size of the box. Notice that at low temperature, it is also possible to obtain precise informations on the fluctuations around the minimizer, see ref. 28.

1.3. Microscopic Model: Bulk Phases

We start by introducing the measures describing finite-range ferromagnetic Ising models with pair interactions in the bulk; the complete description, including the influence of the substrate, is given in Subsection 1.5.

The system is contained in the finite box

$$\mathbb{B}_N \triangleq \{i \in \mathbb{Z}^d : |i_k| \leqslant N, k = 1,\dots, d\}.$$

The interactions are given by a set of real numbers $\mathbf{J} \triangleq (J_{ij})_{i,j \in \mathbb{Z}^d}$, with the following properties:

1. $J_{ij} \triangleq J_{|i-j|}$;
2. the graph $(\mathbb{Z}^d, \mathscr{E}^d_{\mathbf{J}})$ with edge set $\mathscr{E}^d_{\mathbf{J}}$ consisting of all (unoriented) pairs of vertices (i, j) with $J_{|i-j|} > 0$ is connected;
3. $J_k \geqslant 0$ for all $k \in \mathbb{Z}^d$;
4. $J_k = 0$ if $\|k\|_1 > R$, for some $R < \infty$ (the smallest such R is called the range of the interaction).

Let $\bar{\sigma} \in \{-1, 1\}^{\mathbb{Z}^d}$. The Hamiltonian in $\mathbb{B}_N$ with boundary condition $\bar{\sigma}$ is given by

$$\mathbf{H}^{\bar{\sigma}}(\sigma) \triangleq -\tfrac{1}{2} \sum_{(i,j) \subset \mathbb{B}_N} J_{ij}\sigma_i\sigma_j - \sum_{i \in \mathbb{B}_N,\, j \in \mathbb{Z}^d \setminus \mathbb{B}_N} J_{ij}\sigma_i\bar{\sigma}_j,$$

and the corresponding (finite-volume) Gibbs measure at inverse temperature β is the probability measure on $\{-1, 1\}^{\mathbb{Z}^d}$ defined by

$$\mu^{\beta}_{N,\bar{\sigma}}(\sigma) \triangleq \begin{cases} \dfrac{1}{\mathbf{Z}^{\bar{\sigma}}_{\mathbb{B}_N}} \exp(-\beta \mathbf{H}^{\bar{\sigma}}(\sigma)) & \text{if} \quad \sigma_i = \bar{\sigma}_i, \qquad \text{for all} \quad i \notin \mathbb{B}_N, \\ 0 & \text{otherwise.} \end{cases}$$

Two boundary conditions of particular importance are the $+$b.c. defined by $\bar{\sigma} \equiv 1$, and the $-$b.c. $\bar{\sigma} \equiv -1$; the corresponding finite-volume Gibbs states are denoted by $\mu^{\beta}_{N,+}$ and $\mu^{\beta}_{N,-}$ respectively. It follows from correlation inequalities that the weak limits $\mu^{\beta}_{+} \triangleq \lim_{N \to \infty} \mu^{\beta}_{N,+}$ and $\mu^{\beta}_{-} \triangleq \lim_{N \to \infty} \mu^{\beta}_{N,-}$ exist and are translation invariant (in fact ergodic). It is a basic result of Statistical Mechanics that in dimensions $d \geqslant 2$ there exists $0 < \beta_c(d, \mathbf{J}) < \infty$ such that $\mu^{\beta}_{+} = \mu^{\beta}_{-}$ if $\beta < \beta_c$ (this also implies uniqueness of the infinite-volume Gibbs state), while $\mu^{\beta}_{+} \neq \mu^{\beta}_{-}$ if $\beta > \beta_c$. The order parameter associated to this phase transition is the spontaneous magnetization,

$$m^*(\beta) \triangleq \lim_{N \to \infty} \mu^{\beta}_{+}\left(\frac{1}{|\mathbb{B}_N|} \sum_{i \in \mathbb{B}_N} \sigma_i\right) = \mu^{\beta}_{+}(\sigma_0).$$

It can indeed be shown that $m^*(\beta) = 0$ when $\beta < \beta_c$ and $m^*(\beta) > 0$ when $\beta > \beta_c$. Of course, by symmetry, $\mu^{\beta}_{-}(\sigma_0) = -m^*(\beta)$.

1.4. Microscopic Model: Surface Tension

Let $\vec{n} \in \mathbb{S}^{d-1}$ be a unit normal. Assume for definiteness that $(\vec{n}, \vec{e}_d) > 1/\sqrt{d}$. Given two positive real numbers L and M, define

$$\Lambda_{M,L}(\vec{n}) \triangleq \left\{ x \in \mathbb{R}^d : \begin{array}{l} |(x, \vec{e}_k)| \leqslant L/2 \text{ for } k = 1, \dots, d-1 \\ \text{and } |(\vec{n}, x)| \leqslant M/2 \end{array} \right\} \tag{1.4}$$

Thus, $\Lambda_{M,L}(\vec{n})$ is a parallelepiped with the basis orthogonal to $\vec{n}$ and having area $L^{d-1}/(\vec{n}, \vec{e}_d)$ and height M. With a slight abuse of notation we identify $\Lambda_{M,L}(\vec{n})$ with its intersection with $\mathbb{Z}^d$; $\Lambda_{M,L}(\vec{n}) = \Lambda_{M,L}(\vec{n}) \cap \mathbb{Z}^d$. Let $\mathbf{Z}^{+}_{\Lambda_{M,L}(\vec{n})}$ and $\mathbf{Z}^{\pm}_{\Lambda_{M,L}(\vec{n})}$ be the partition functions on $\Lambda_{M,L}(\vec{n})$ with respectively "$+$" and $\vec{n}$ boundary conditions, the latter being defined by $\sigma_i = \operatorname{sign}((\vec{n}, i))$, with sign(0) = 1. The (per unit area) surface tension of the $\pm$-interface stretched in the direction orthogonal to $\vec{n}$ is defined via:

$$\tau_\beta(\vec{n}) \triangleq -\lim_{\substack{L\to\infty \\ M\to\infty}} \frac{(\vec{n}, \vec{e}_d)}{L^{d-1}} \log \frac{\mathbf{Z}^{\pm}_{\Lambda_{M,L}(\vec{n})}}{\mathbf{Z}^{+}_{\Lambda_{M,L}(\vec{n})}} \tag{1.5}$$

The important ferromagnetic feature of the model (or, possibly, rather of the proof of the corresponding fact) is that the limit in (1.5) is defined and *does not* depend on the order in which the numbers L and M go to infinity. In ref. 29 this result has been formulated for the sequences of domains tending to ∞ in the sense of Landford, but the corresponding proof goes through also in the $\lim_{M\to\infty} \lim_{L\to\infty}$ case.

Finally, as it has been proven in ref. 30, the surface tension is uniformly strictly positive whenever $\beta > \beta_c$.

1.5. Microscopic Model: Wall Free Energy

Microscopic models for the substrate surface have been studied in refs. 31 and 32 in the context of the nearest neighbor model. Let us reformulate, with brief comments on the validity in the general case we consider here, all those of the results of ref. 32 which we are going to use in the sequel.

Define the lattice half-space $\mathbb{L}^d \triangleq \{i \in \mathbb{Z}^d : i_d > 0\}$. Our spin configurations σ are the elements of $\{-1, 1\}^{\mathbb{L}^d}$. The microscopic influence of the substrate is modeled by magnetic fields (chemical potentials in the lattice gas language) $\eta = (\eta_1,..., \eta_r)$ acting on spins in the first r microscopic layers[5] of $\mathbb{L}^d$. Thus, the formal Hamiltonian on $\{-1, 1\}^{\mathbb{L}^d}$ is given by:

$$\mathbf{H}^\eta(\sigma) \triangleq -\sum_{(i,j)\subset\mathbb{L}^d} J_{ij}\sigma_i\sigma_j - \sum_{k=1}^{r} \sum_{i\in\mathbb{L}^d : i_d = k} \eta_k \sigma_i. \tag{1.6}$$

Given two different vectors η and η' of boundary magnetic fields we say that η is larger than η' if $\eta_k \geqslant \eta'_k$ for all $k = 1,..., r$.

In order to describe the appropriate surface energy corrections induced by the substrate, we put it in contact with either of the two "$-$" or "$+$" bulk phases. This is done in a standard way: Given $N \in \mathbb{N}$ we construct the finite lattice box

$$\mathbb{D}_N \triangleq \{i \in \mathbb{L}^d : -N < i_k \leqslant N, k = 1,..., d-1, 0 < i_d \leqslant N\},$$

[5] We do not assume any connection between the number r and the range of the interaction $\{J_{ij}\}$.

and consider the models with formal Hamiltonian (1.6) and, respectively, with "$-$" and "$+$" boundary conditions on $\mathbb{L}^d \setminus \mathbb{D}_N$. Let $\mu^{\beta,\eta}_{\mathbb{D}_N,-}$ and $\mu^{\beta,\eta}_{\mathbb{D}_N,+}$ be the corresponding finite volume Gibbs states on $\{-1,1\}^{\mathbb{D}_N}$ and $\mathbf{Z}^{\beta,\eta}_{\mathbb{D}_N,-}$, $\mathbf{Z}^{\beta,\eta}_{\mathbb{D}_N,+}$ be the associated partition functions. The difference $\Delta_{\beta,\eta}$ between the interfacial free energies is defined via:

$$\Delta_{\beta,\eta} \triangleq \lim_{N\to\infty} \frac{1}{(2N)^{d-1}} \log \frac{\mathbf{Z}^{\beta,\eta}_{\mathbb{D}_N,+}}{\mathbf{Z}^{\beta,\eta}_{\mathbb{D}_N,-}}. \tag{1.7}$$

Proposition 1.1.[32] The limit in (1.7) is well defined and monotone non-increasing in η. Furthermore, uniformly in η

$$|\Delta_{\beta,\eta}| \leqslant \tau_\beta(\vec{e}_d). \tag{1.8}$$

As in ref. 32 the assertion of the theorem follows from the FKG properties of the ferromagnetic measures. We refer to Section 3.1 in ref. 32 (with obvious modifications to fit the general case we consider here) for details.

Remark 1.1. On the heuristic level the bound in (1.8) should be clear: one possible scenario under the $\mu^{\beta,\eta}_{\mathbb{D}_N,-}$ measure is to create a microscopically divergent film of "$+$" phase along the substrate surface reducing, thus, the ratio in (1.7) to the one appearing in the definition of $\tau_\beta(\vec{e}_d)$ in (1.5). This scenario certainly becomes dominant at large positive values of η, which happens to be equivalent to the uniqueness of the limiting boundary Gibbs state.[32] A non-trivial analysis of the corresponding phase diagram (in terms of η) is, however, an almost entirely open question, c.f. refs. 33–35 for the related results for effective interface models.

1.6. The Result

In principle, the results of the paper should hold for any $\beta > \beta_c$. However, our approach relies on the validity of Pisztora's renormalization machinery, which we shall describe in detail in Section 3. Consequently, we need the following assumption on the inverse temperature β:

(A) Pisztora's coarse graining holds at the inverse temperature β.

We use $\mathfrak{B}$ to denote the set of $\beta > \beta_c$ for which the above assumption holds.

Define the total average magnetization on $\mathbb{D}_N$ as

$$\mathbf{M}_{\mathbb{D}_N}(\sigma) \triangleq \frac{1}{|\mathbb{D}_N|} \sum_{i \in \mathbb{D}_N} \sigma_i .$$

The averaged magnetization $\mathbf{M}_{\mathbb{D}_N}$ concentrates (as $N \to \infty$) under the measure $\mu^{\beta,\eta}_{\mathbb{D}_N,+}$ on the spontaneous "+" phase magnetization $m^*(\beta)$. In other words the bulk phase induced by $\mu^{\beta,\eta}_{\mathbb{D}_N,+}$ is the "+" phase. In order to enforce phase segregation one should introduce a canonical type constraint which would shift the average magnetization $\mathbf{M}_{\mathbb{D}_N}$ inside the phase coexistence region. In the lattice gas language this would amount to fixing the average density of particles strictly between the two extremal equilibrium densities. The canonical constraints we are working with here are of the integral type: Given $m \in (-m^*(\beta), m^*(\beta))$, let us define the canonical conditional measure

$$\begin{aligned}\mu^{\beta,\eta}_{\mathbb{D}_N,m,+}(\sigma) &= \frac{1}{\mathbf{Z}^{\beta,\eta}_{\mathbb{D}_N,m,+}} \exp[-\beta \mathbf{H}^{\eta}_{\mathbb{D}_N}(\sigma)]\, \mathbb{1}_{\{\mathbf{M}_{\mathbb{D}_N} \leqslant m\}}(\sigma) \\ &= \mu^{\beta,\eta}_{\mathbb{D}_N,+}(\sigma \mid \mathbf{M}_{\mathbb{D}_N} \leqslant m).\end{aligned} \tag{1.9}$$

The impact of such a conditioning on the bulk properties should amount to the creation of a "−" phase island of macroscopic size close to

$$v(m) \triangleq \frac{m^* - m}{2m^*}. \tag{1.10}$$

Our microscopic justification of the Winterbottom construction above the complete wetting threshold; $\Delta_{\beta,\eta} > -\tau_\beta(\vec{e}_d)$, gives a rigorous meaning to the heuristic picture above and, moreover, asserts that under the natural scaling the shape of the "−" phase island converges to the Winterbottom shape $\mathscr{K}_{\beta,\eta}(v(m))$. Such a result is, of course, impossible unless $\mathscr{K}_{\beta,\eta}(v(m))$ fits inside the unit box (the macroscopic vessel of the system)

$$\hat{\mathbb{D}} \triangleq [-\tfrac{1}{2}, \tfrac{1}{2}]^{d-1} \times [0, 1],$$

which is obtained by the scaling of $\mathbb{D}_N$ by the factor $1/(2N)$. Define

$$\bar{m}(\beta, \eta) = \min\{m : \mathscr{K}_{\beta,\eta}(v(m)) \subseteq \hat{\mathbb{D}}\}.$$

Theorem 1.1. Assume that $\beta \in \mathfrak{B}$ and all the components of the boundary magnetic field $\eta \in \mathbb{R}^r$ have the same sign. Assume, furthermore, that $\Delta_{\beta,\eta} > -\tau_\beta(\vec{e}_d)$. Then, for every $m \in \,]\bar{m}(\beta,\eta), m^*(\beta)[$,

$$\lim_{N\to\infty} \frac{1}{N^{d-1}} \log \frac{\mathbf{Z}^{\beta,\eta}_{\mathbb{D}_N, m, +}}{\mathbf{Z}^{\beta,\eta}_{\mathbb{D}_N, +}} = \lim_{N\to\infty} \frac{1}{N^{d-1}} \log \mu^{\beta,\eta}_{\mathbb{D}_N, +}(\mathbf{M}_{\mathbb{D}_N} < m)$$
$$= -\mathscr{W}_{\beta,\eta}(\mathscr{K}_{\beta,\eta}(v(m))) \triangleq -w^{\star}_{\beta,\eta}(m). \quad (1.11)$$

Theorem 1.1 gives a microscopic justification of the thermodynamical variational problem leading to the Winterbottom construction, but, as it is, yields little information on the statistical properties of the microscopic spin configuration under the canonical measure $\mu^{\beta,\eta}_{\mathbb{D}_N, m, +}$. In the $\mathbb{L}_1$-approach we pursue here the microscopic spin fields are identified only on the (renormalized) mesoscopic level through the local order parameters or, equivalently, through locally averaged magnetization profiles. These local order parameters are piece-wise constant functions on the continuum box $\hat{\mathbb{D}}$ and they are constructed from the microscopic spin configuration $\sigma \in \{-1, 1\}^{\mathbb{D}_N}$ in the following way:

Choice of Scales. We shall always take the microscopic size of the binary form, $N = 2^n$. Similarly, all the intermediate mesoscopic scales are of the form $K = 2^k$, $k \in \mathbb{N}$.

Partition of $\mathbb{D}_N$. At each fixed mesoscopic scale $K = 2^k$ we split the microscopic vessel $\mathbb{D}_N$ into the disjoint union of shifts of the mesoscopic box $\mathbb{B}_K \triangleq \{-\frac{1}{2}K,..., \frac{1}{2}K\}^d$. These shifted boxes are centered at the lattice points from the rescaled set $\mathbb{D}_{N,K} \triangleq K(\mathbb{D}_{N/K} - (1/2,..., 1/2))$:

$$\mathbb{D}_N = \bigvee_{i \in \mathbb{D}_{N,K}} \mathbb{B}_K(i), \quad (1.12)$$

where $\mathbb{B}_K(i) \triangleq i + \mathbb{B}_K$.

Induced Partition of $\hat{\mathbb{D}}$. We scale (1.12) by the factor N. With each mesoscopic lattice box $\mathbb{B}_K(i)$ in the partition (1.12) we associate the continuum box

$$\hat{\mathbb{B}}_{N,K}(x) \triangleq x + \left(-\frac{1}{2}\frac{K}{N}, \frac{1}{2}\frac{K}{N}\right]^d$$

centered at the point $x = \frac{1}{N} i$. We use $\hat{\mathbb{D}}_{N,K}$ to denote the set of all such centers x; $\hat{\mathbb{D}}_{N,K} = \frac{1}{N}\mathbb{D}_{N/K}$. The induced mesoscopic partition of $\hat{\mathbb{D}}$ is given by:

$$\hat{\mathbb{D}} = \text{closure}\left(\bigvee_{x \in \hat{\mathbb{D}}_{N,K}} \hat{\mathbb{B}}_{N,K}(x)\right). \quad (1.13)$$

The local magnetization profile $\mathscr{M}_{N,K}$ corresponding to a microscopic spin configuration $\sigma \in \{-1, 1\}^{\mathbb{D}_N}$ is a function on the continuum box $\hat{\mathbb{D}}$, piece-wise constant with respect to the partition (1.13). For every $x \in \widehat{\mathbb{D}}_{N,K}$ the value of $\mathscr{M}_{N,K}$ on $\widehat{\mathbb{B}}_{N,K}(x)$ equals to the averaged magnetization on the corresponding lattice box $\mathbb{B}_K(i)$ with $i = Nx$;

$$\mathscr{M}_{N,K}(\sigma)(y) = \frac{1}{K^d} \sum_{j \in \mathbb{B}_K(i)} \sigma_j, \qquad \text{for} \quad y \in \widehat{\mathbb{B}}_K(x).$$

Informally, $\mathscr{M}_{N,K}(\sigma)$ is the resolution with which one observes the microscopic spin field σ. Local proximity of the Gibbs measure to one of the pure phases is, in this way, quantified by the proximity of $\mathscr{M}_{N,K}(\sigma)$ to the corresponding order parameter $\pm m^*(\beta)$. Our probabilistic counterpart of the thermodynamic Theorem 1.1 states that on coarser scales local magnetization profiles $\mathscr{M}_{N,K}$ comply, with the overwhelming conditional probability $\mu^{\beta,\eta}_{\mathbb{D}_N,m,+}$, with the thermodynamic prediction. This agreement has to be understood in the $\mathbb{L}_1$-sense: Given a measurable subset $A \subset \mathbb{R}^d$ let us define the function $\mathbb{1}_A \in \mathbb{L}_1(\widehat{\mathbb{D}})$ via:

$$\mathbb{1}_A(x) = \begin{cases} +1, & \text{if} \quad x \in A \\ -1, & \text{if} \quad x \notin A \end{cases}$$

Theorem 1.2. Fix a number $\nu < 1/d$ and assume that $\beta \in \mathfrak{B}$ and all the components of the boundary magnetic field $\eta \in \mathbb{R}^r$ have the same sign. Assume, furthermore, that $\Delta_{\beta,\eta} > -\tau_\beta(\vec{e}_d)$. Let $m \in]\bar{m}(\beta,\eta), m^*(\beta)[$. Then for every $\delta > 0$ fixed, one can choose a finite mesoscopic scale $K_0 = K_0(\beta,\eta,\delta)$, such that

$$\lim_{N\to\infty} \min_{K_0 \leqslant K \leqslant N^\nu} \mu^{\beta,\eta}_{\mathbb{D}_N,m,+}\left(\min_x \|\mathscr{M}_{N,K}(\sigma) - m^*_\beta \mathbb{1}_{\{x+\mathscr{K}_{\beta,\eta}(v(m))\}}\|_{\mathbb{L}_1(\widehat{\mathbb{D}})} \leqslant \delta \right) = 1.$$

Of course, the above minimum can be restricted to the shifts of $\mathscr{K}_{\beta,\eta}(v(m))$ along the wall $\mathscr{P}_d$ in the partial wetting/drying case; $|\Delta_{\beta,\eta}| < \tau_\beta(\vec{e}_d)$, and, respectively, to all admissible shifts of the Wulff shape $\mathscr{K}_\beta(v)$ within $\widehat{\mathbb{D}}$ in the case of complete drying; $\Delta_{\beta,\eta} = \tau_\beta(\vec{e}_d)$.

1.7. Structure of the Paper and Further Remarks

To a large extent the proof of Theorems 1.1 and 1.2 is a book-keeping exercise based on the renormalization formalism we have developed for the nearest-neighbor Ising model in ref. 12. In that paper we tried to decouple deep model oriented facts from the relatively soft $\mathbb{L}_1$ techniques. The fact that the proof swiftly goes through in the general case we consider here is a dividend of such an approach.

The deep model oriented fact is the validity of Pisztora's coarse graining[3] and its application to the relaxation properties of the FK measures;[10] both have been originally developed in the nearest neighbor context, but go through literally without changes in any model with translation invariant finite-range ferromagnetic pair interactions. In Section 2 we briefly recall the corresponding construction and show how to adjust it to treat the boundary surface tension. In Section 3 we set up the machinery of random mesoscopic phase labels and relate it to the context of the theory of functions of bounded variation. Since the boundary field introduces a singularity of the surface tension at the bottom wall, some additional care (as compared to ref. 12) is needed at this stage. Furthermore, the appropriate FK-representation in the case of negative boundary magnetic fields happens to be a conditional one. In particular, the compactness estimates have to be modified.

The proof of Theorem 1.2 (and, on the way, Theorem 1.1) is relegated to Section 4. In view of the results and adjustments of the preceding sections it closely follows the pattern laid down in ref. 12. Accordingly, our exposition in this section will be rather concise with repeated references to the appropriate parts of ref. 12.

Finally, we would like to point out that though the minimizers of the variational problem $\mathbf{(WP)}_{\beta,\eta}$ restricted to the sets $P \subseteq \widehat{\mathbb{D}}$ are, in general, not known for $m < \bar{m}_{\beta,\eta}$, the $\mathbb{L}_1$-approach still could be pushed through to yield meaningful results in the sense that one can prove that the mesoscopic configurations concentrate around a (in general unknown) set of surface energy minimizers. Such an idea of decoupling between the probabilistic analysis of the microscopic model and the investigation of the limiting variational problem has been put forward in ref. 27 and, recently, systematically exploited in ref. 11. Indeed, the microscopic derivation of phase coexistence is based upon local decoupling estimates and on surgical procedures in small regions localized along the interface of the crystals. Therefore, from the probabilist point of view the approach is local and does not rely on the global shape of the solutions to the corresponding variational problems. On the technical level, however, one should adjust the approximation results of Subsection 4.1 to the case of unknown minimizers. The approximation procedure we use here hinges on the fact that the minimizers are known to be convex. If the surface tension is singular at the boundary (that is in the partial wetting/drying cases), a complete geometric analysis without such an information would require a serious detour into the geometric measure theory which, from our point of view, would not bring any deeper insights on the microscopic phenomena involved in the Winterbottom construction. We refer to ref. 8 or to ref. 11 for a detailed exposition of the appropriate facts from the theory of functions of bounded variation.

A more challenging problem is to gain a better understanding of the statistical properties of phase boundaries, even on mesoscopic or macroscopic scales. For example, the only information the $\mathbb{L}_1$-approach yields on the entropically repulsed interface is a localization on the macroscopic scale, which is, of course, not such a fantastic result, since we absolutely do not control the typical microscopic size of the corresponding fluctuations.

2. FK REPRESENTATION

FK representation is an artificial coupling between the ferromagnetic spin models and the, so called, random cluster measures. Its power rests with the fact that the latter are in the *uniqueness* regime and possess uniform decoupling and relaxation properties, even when the original spin system is in the phase coexistence region.[6] The art of the FK representation is, therefore, to single out such spin configuration properties which admit a reformulation in the FK language.

2.1. Representation of the Bulk States

With a given finite range translation invariant pair interaction potential $\mathbf{J} = \{J_{ij} = J_{|i-j|}\}$ we associate the graph $(\mathbb{Z}^d, \mathscr{E}^d_{\mathbf{J}})$ where the edge set $\mathscr{E}^d_{\mathbf{J}}$ consists of all (unoriented) pairs of vertices (i, j) with $J_{|i-j|} > 0$. Each such pair will be called a bond of $\mathscr{E}^d_{\mathbf{J}}$. The set $\Omega_{\mathbf{J}} \triangleq \{0, 1\}^{\mathscr{E}^d_{\mathbf{J}}}$ is the sample space for the dependent percolation measures associated with $\mathbf{J}$. Given $\omega \in \Omega_{\mathbf{J}}$ and a bond $b = (i, j) \in \mathscr{E}^d_{\mathbf{J}}$, we say that b is open if $\omega(b) = 1$. Two sites of $\mathbb{Z}^d$ are said to be connected if one can be reached from another via a chain of open bonds. Thus, each $\omega \in \Omega$ splits $\mathbb{Z}^d$ into the disjoint union of maximal connected components, which are called the open clusters of Ω. Given a finite subset $B \subset \mathbb{Z}^d$ we use $c_B(\omega)$ to denote the number of different open finite clusters of ω which have a non-empty intersection with B.

We define next the FK measures which correspond to the finite volume spin Gibbs states on the boxes $\mathbb{B}_N$: First of all these measures put non-trivial weights on the percolation configurations $\omega \in \Omega_{N,\mathbf{J}} \triangleq \{0, 1\}^{\mathscr{E}^d_{N,\mathbf{J}}}$, where the set $\mathscr{E}^d_{N,\mathbf{J}}$ comprises those of the bonds $b \in \mathscr{E}_{\mathbf{J}}$ which intersect $\mathbb{B}_N$. The boundary conditions are specified by a frozen percolation configuration $\pi \in \Omega^c_{N,\mathbf{J}} \triangleq \Omega_{\mathbf{J}} \setminus \Omega_{N,\mathbf{J}}$. Using the shortcut $c^\pi_N(\omega) = c_{\mathbb{B}_N}(\omega \vee \pi)$ for the joint

[6] Strictly speaking, this is known to be true for all sub-critical temperatures, except possibly for a countable subset of the latter, see Subsection 3.2.

configuration $\omega \vee \pi \in \mathscr{E}_{\mathbf{J}}$, we define the finite volume FK measure $\Phi_N^{\beta,\pi}$ on $\Omega_{N,\mathbf{J}}$ with the boundary conditions π as:

$$\Phi_N^{\beta,\pi}(\omega) \triangleq \frac{1}{\mathbf{Z}_N^{\beta,\pi}} \left(\prod_{b \in \mathscr{E}_{N,\mathbf{J}}} (1-p_\beta(b))^{1-\omega_b} (p_\beta(b))^{\omega_b} \right) 2^{c_N^\pi(\omega)}, \tag{2.1}$$

where, for a bond $b=(i,j) \in \mathscr{E}_{N,\mathbf{J}}$, we define the corresponding percolation probability $p_\beta(b) = 1 - \exp(-2\beta J_{|i-j|})$.

The measures $\Phi_N^{\beta,\pi}$ are FKG partially ordered with respect to the natural partial order of the boundary condition π. Thus, the extremal ones correspond to the free ($\pi \equiv 0$) and wired ($\pi \equiv 1$) boundary conditions and are denoted as $\Phi_N^{\beta,\mathrm{f}}$ and $\Phi_N^{\beta,\mathrm{w}}$ respectively. The corresponding infinite volume ($N \to \infty$) limits $\Phi^{\beta,\mathrm{f}}$ and $\Phi^{\beta,\mathrm{w}}$ always exist, and it takes a relatively soft argument[36] to show that for all but at most a countable set of temperatures $\beta \in \mathbb{R}_+$ there is a unique infinite volume FK-measure:

(A1) We assume that the inverse temperature β satisfies $\Phi^{\beta,\mathrm{f}} = \Phi^{\beta,\mathrm{w}}$.

Assumption **(A)** is a combination of **(A1)** and the condition that there is percolation in slabs, see Assumption **(A2)** below.

The spontaneous magnetization $m^*(\beta)$ admits the following expression in FK terms:

$$m^*(\beta) = \lim_{N\to\infty} \Phi_N^{\beta,\mathrm{w}}(0 \leftrightarrow \mathbb{B}_N^c) = \Phi^{\beta,\mathrm{w}}(0 \leftrightarrow \infty). \tag{2.2}$$

More generally, the finite volume (spin) Gibbs state $\mu_{N,+}^\beta$ on $\{-1,1\}^{\mathbb{B}_N}$ (see Subsection 1.3) can be recovered[37] from the wired FK-state $\Phi_N^{\beta,\mathrm{w}}$ as follows: Sample a percolation configuration $\omega \in \Omega_{N,\mathbf{J}}$ from $\Phi_N^{\beta,\mathrm{w}}$. Spins at the sites of $\mathbb{B}_N$ which belong to the wired component of $\omega \vee \mathbf{1}$ (that is connected to the boundary $\mathbb{B}_N^c$) are assigned value $+1$, whereas the remaining open clusters of $\omega \vee \mathbf{1}$ are painted to ± 1 with probability $1/2$ each.

2.2. Representation of the Surface Tension

The surface tension of the spin model (1.5) can be expressed in the FK language in the following way: Let us split the boundary $\partial \Lambda_{M,L}(\vec{n}) = \mathbb{Z}^d \setminus \Lambda_{M,L}(\vec{n})$ into two pieces:

$$\begin{aligned}\partial \Lambda_{M,L}(\vec{n}) &= \partial^+ \Lambda_{M,L}(\vec{n}) \cup \partial^- \Lambda_{M,L}(\vec{n}) \\ &\triangleq (\partial \Lambda_{M,L}(\vec{n}) \cap \{i \in \mathbb{Z}^d : i_d > 0\}) \cup (\partial \Lambda_{M,L}(\vec{n}) \cap \{i \in \mathbb{Z}^d : i_d \leqslant 0\})\end{aligned}$$

Then the ratio of partition functions in (1.5) is equal to

$$\frac{\mathbf{Z}^{\pm}_{\Lambda_{M,L}(\vec{n})}}{\mathbf{Z}^{+}_{\Lambda_{M,L}(\vec{n})}} = \Phi^{\beta,\mathrm{w}}_{\Lambda_{M,L}(\vec{n})}\big(\partial^+\Lambda_{M,L}(\vec{n}) \nleftrightarrow \partial^-\Lambda_{M,L}(\vec{n})\big), \tag{2.3}$$

where the FK-percolation event $\{\partial^+\Lambda_{M,L}(\vec{n}) \nleftrightarrow \partial^-\Lambda_{M,L}(\vec{n})\}$ means that no site of $\Lambda_{M,L}(\vec{n})$ is connected both to $\partial^+\Lambda_{M,L}(\vec{n})$ and to $\partial^-\Lambda_{M,L}(\vec{n})$.

The expression (2.3) for the surface tension is not very convenient in practice. Indeed, in higher dimensions one should not expect the interface to decouple along microscopically wired b.c. It would therefore be very useful to have a more robust definition which would still fit in the framework of the $\mathbb{L}_1$-theory. The idea is twofold. First, one can relax the pinning of the "microscopic interface" along the lateral sides of $\Lambda_{M,L}(\vec{n})$, by requesting only that the top and bottom faces of the box are disconnected. Of course in order to still recover the surface tension in the direction orthogonal to $\vec{n}$, one has to choose $M \ll L$. The control of the interactions on the lateral sides is one of the major technical contribution by Cerf in ref. 8. The second point is to relax boundary conditions, i.e., to replace the wired b.c. by some arbitrary π. In order to avoid a possible deformation of the "microscopic interface" due to the attraction by some b.c. π, one has to let enough room for the system to relax to its unique equilibrium phase. This is done by imposing the boundary condition π outside the bigger box $\Lambda_{L,L}$.

More precisely, let $\partial^{\mathrm{top}}\Lambda_{M,L}(\vec{n})$, resp. $\partial^{\mathrm{bot}}\Lambda_{M,L}(\vec{n})$, be the face of $\partial^+\Lambda_{M,L}(\vec{n})$, resp. $\partial^-\Lambda_{M,L}(\vec{n})$, normal to $\vec{n}$. As a consequence of the relaxation properties of FK measures derived by Cerf and Pisztora[10] (Proposition 3.1), the following holds

Lemma 2.1. Let $\varepsilon > 0$. For any $\beta \in \mathfrak{B}$,

$$\tau_\beta(\vec{n}) = -\frac{(\vec{n}, \vec{e}_d)}{L^{d-1}} \log \Phi^{\beta,\pi}_{\Lambda_{L,L}(\vec{n})}\big(\partial^{\mathrm{top}}\Lambda_{\varepsilon L,L}(\vec{n}) \nleftrightarrow \partial^{\mathrm{bot}}\Lambda_{\varepsilon L,L}(\vec{n})\big) + c_{\varepsilon,L}(\pi,\vec{n}),$$

with $c_{\varepsilon,L}(\pi,\vec{n})$ going to zero uniformly in π and $\vec{n}$, as L goes to infinity and ε goes to zero.

2.3. Representation of the Boundary States

The FK-notation for the boundary Gibbs states closely follows the setup introduced in the previous subsection. The relevant graph for the inter-spin interactions is $(\mathbb{L}^d, \mathscr{L}^d_{\mathrm{J}})$ which is just the restriction of $(\mathbb{Z}^d, \mathscr{E}^d_{\mathrm{J}})$ to the half-space $\mathbb{L}^d$. In order to incorporate the boundary magnetic field

$\eta=(\eta_1,\ldots,\eta_r)$ (see (1.6)) we augment this graph with a ghost site $\mathfrak{g}$ connected to all the sites in the first r layers of $\mathbb{L}^d$. Thus the edge set for the boundary model is given by

$$\mathscr{L}^d_{\mathbf{J},\eta} \triangleq \mathscr{L}^d_{\mathbf{J}} \cup \{(i,\mathfrak{g}) \mid i\in\mathbb{L}^d \text{ and } i_d \leqslant r\}.$$

Similarly, given $N\in\mathbb{N}$, we use $\mathscr{L}^d_{N,\mathbf{J},\eta}$ to denote the set of bonds of $\mathscr{L}^d_{\mathbf{J},\eta}$ which have a non-empty intersection with $\mathbb{D}_N$. The sample space for finite volume FK states on $\mathbb{D}_N$ is given by

$$\Xi_{N,\mathbf{J},\eta} \triangleq \{0,1\}^{\mathscr{L}^d_{N,\mathbf{J},\eta}}.$$

Assume that $\eta\geqslant 0$ (that is $\eta_k\geqslant 0$ for $k=1,\ldots,r$). Given a configuration $\pi\in\{0,1\}^{\mathscr{L}^d_{\mathbf{J},\eta}\setminus\mathscr{L}^d_{N,\mathbf{J},\eta}}$, the FK measure $\Phi^{\beta,\eta}_N$ on $\Xi_{N,\mathbf{J},\eta}$ is defined by

$$\Phi^{\beta,\eta,\pi}_N(\xi)=\frac{1}{\mathbf{Z}^{\beta,\eta}_N}\left(\prod_{b\in\mathscr{L}^d_{N,\mathbf{J},\eta}}(1-p_\beta,\eta(b))^{1-\xi_b}\,(p_\beta,\eta(b))^{\xi_b}\right)2^{c^\pi_N(\xi)}, \tag{2.4}$$

where $c^\pi_N(\xi)$ denotes the number of open finite clusters of $\xi\vee\pi$ which intersect $\mathbb{D}_N$ and do not contain the ghost site $\mathfrak{g}$, whereas the percolation probabilities $p_\beta,\eta(b)$ are defined exactly as in (2.1) for the bonds $b\in\mathscr{L}^d$ and equal to $1-\mathrm{e}^{-2\beta\eta_k}$ on the ghost bonds $b=(i,\mathfrak{g})$ with $i_d=k$, $k=1,\ldots,r$. We suppress the super-index π for the wired state with the boundary conditions $\pi\equiv 1$.

The boundary Gibbs state $\mu^{\beta,\eta}_{\mathbb{D}_N,+}$ can be reconstructed as follows: sample a bond configuration $\xi\in\Xi_{N,\mathbf{J},\eta}$ from $\Phi^{\beta,\eta}_N$, and paint with 1 all the clusters of ξ connected either to $\mathbb{L}^d\setminus\mathbb{D}_N$ or to $\mathfrak{g}$, whereas all the remaining clusters of ξ are to be painted into ± 1 with probability $1/2$ each. The corresponding joint bond-spin probability measure is denoted by $\mathbb{P}^{\beta,\eta}_N$.

Remark 2.1. The FK measures $\Phi^{\beta,\eta,\pi}_N$ corresponding to the non-negative magnetic fields $\eta\geqslant 0$ are the basic ones and we shall use them in all our considerations.

The representation derived in the case of non-positive magnetic fields $\eta=(-|\eta_1|,\ldots,-|\eta_r|)$ can be described as follows: Define the FK percolation event $\mathfrak{J}_N\subset\Xi_{N,\mathbf{J},\eta}$ as

$$\mathfrak{J}_N \triangleq \{\xi\in\Xi_{N,\mathbf{J},\eta} \mid \mathfrak{g}\nleftrightarrow\mathbb{L}^d\setminus\mathbb{D}_N\}, \tag{2.5}$$

and set $\Phi_N^{\beta,\eta}(\cdot) = \Phi_N^{\beta,|\eta|}(\cdot \mid \mathfrak{J}_N)$. Then, the boundary Gibbs measure $\mu^{\beta,\eta}_{\mathbb{D}_N,+}$ can be reconstructed from $\Phi_N^{\beta,\eta}(\cdot)$ as above, except that the spins connected to $\mathfrak{g}$ are, this time, painted into -1. The corresponding joint measure is denoted by $\mathbb{P}^{\beta,|\eta|}_{N,\pm}$. With a slight abuse of notation we shall write (keeping in mind that the $\mathfrak{g}$-cluster is repainted into -1):

$$\mathbb{P}^{\beta,|\eta|}_{N,\pm}(\cdot) = \mathbb{P}^{\beta,|\eta|}_{N}(\cdot \mid \mathfrak{J}_N). \tag{2.6}$$

2.4. Representation of the Wall Free Energy

In view of the spin-flip symmetries the difference $\Delta_{\beta,\eta}$ (see (1.7)) admits the following expression in FK terms (recall the convention $\eta \geqslant 0$):

$$\Delta_{\beta,\eta} = -\lim_{N\to\infty} \frac{1}{(2N)^{d-1}} \log \Phi_N^{\beta,\eta}(\mathfrak{J}_N), \tag{2.7}$$

and, accordingly, if η is a negative boundary field $\Delta_{\beta,\eta} = -\Delta_{\beta,|\eta|}$.

As in the case of the surface tension, the above definition is too restrictive to be used in practice, because of the stringent condition of microscopically wired b.c. Fortunately, a more robust expression is also available here.

Let $\partial^{\mathrm{top},\varepsilon}\mathbb{D}_N \triangleq \mathbb{D}_N \cap \{i \in \mathbb{L}^d : i_d = [\varepsilon N]\}$, where $[x]$ denotes the integer part of $x \in \mathbb{R}^d$. Then, using the relaxation properties of FK measures [Ref. 10, Proposition 3.1], one can prove the following

Lemma 2.2. Let $\varepsilon > 0$. For any $\beta \in \mathfrak{B}$ and any $\eta \geqslant 0$,

$$\Delta_{\beta,\eta} = -\frac{1}{(2N)^{d-1}} \log \Phi_N^{\beta,\eta,\pi}(\mathfrak{g} \nleftrightarrow \partial^{\mathrm{top},\varepsilon}\mathbb{D}_N) + c_{\varepsilon,N}(\pi),$$

with $c_{\varepsilon,N}(\pi)$ going to zero uniformly in π, as N goes to infinity and ε goes to zero.

3. RENORMALIZATION

In order to perform the analysis of phase coexistence in a macroscopic setting, a convenient formalism, the Geometric Measure Theory, is introduced in Subsection 3.1. The embedding of the discrete spin system in the continuum setting relies on the introduction of renormalized variables, the mesoscopic phase labels (Subsection 3.2).

3.1. Functions of Bounded Variation

We refer the reader to ref. 38 for an introduction to the theory of functions of bounded variation and to ref. 12 for a discussion on its relevance in the context of phase coexistence.

Let $\mathcal{O}$ be an open smooth neighborhood of $\hat{\mathbb{D}}$. On the macroscopic scale the system is characterized by the boundary condition $g \in \mathrm{BV}(\mathcal{O} \setminus \hat{\mathbb{D}}, \{\pm 1\})$ and by a ± 1 phase function $u \in \mathrm{BV}(\mathrm{int}\, \hat{\mathbb{D}}, \{\pm 1\})$. The fact that $u(x) = 1$ for some $x \in \mathrm{int}\, \hat{\mathbb{D}}$ means that locally at x the system is in equilibrium in the "+" phase. Thus, u should be interpreted as a signed indicator of the regions containing the different phases and the boundary of the set $\{u \vee g = -1\}$, where

$$u \vee g(x) = \begin{cases} u(x) & \text{if} \quad x \in \mathrm{int}\, \hat{\mathbb{D}} \\ g(x) & \text{if} \quad x \in \mathcal{O} \setminus \hat{\mathbb{D}} \end{cases} \tag{3.1}$$

as the interface in the presence of g-boundary condition.

It is well known[38] that $u \vee g \in \mathrm{BV}(\mathcal{O}, \{\pm 1\})$ whenever the phase function $u \in \mathrm{BV}(\mathrm{int}\, \hat{\mathbb{D}}, \{\pm 1\})$. Now, for any v in $\mathrm{BV}(\mathcal{O}, \{\pm 1\})$, there exists a generalized notion of the boundary of $\{v = -1\}$ called reduced boundary and denoted by $\partial^* v$. If $\{v = -1\}$ is a regular set, $\partial^* v$ coincides with the usual boundary ∂v. Given a boundary condition $g \in \mathrm{BV}(\mathcal{O} \setminus \hat{\mathbb{D}}, \{\pm 1\})$ and a phase function $u \in \mathrm{BV}(\mathrm{int}\, \hat{\mathbb{D}}, \{\pm 1\})$ we use $\partial^*_g u$ to denote the reduced boundary of u in the presence of the b.c. g:

$$\partial^*_g u = \partial^*(u \vee g) \cap \hat{\mathbb{D}} = \partial^*(u \vee g) \setminus \partial^* g. \tag{3.2}$$

In the setting we are working with here, there are two natural types of boundary conditions depending on the sign of the magnetic field either $g = g_+ \equiv 1$ in the case of positive magnetic fields $\eta \geqslant 0$, or

$$g(x) = g_\pm(x) = \begin{cases} +1, & \text{if} \quad x_d > 0 \\ -1, & \text{if} \quad x_d \leqslant 0 \end{cases} \tag{3.3}$$

in the case of negative magnetic fields $\eta < 0$.

For any non-negative boundary magnetic field η and any $g \in \mathrm{BV}(\mathcal{O} \setminus \hat{\mathbb{D}}, \{\pm 1\})$ define the functional

$$\mathcal{W}_{\beta,\eta}(u \mid g) = \int_{\partial^*_g u \setminus \mathcal{P}_d} \tau_\beta(\vec{n}_x)\, d\mathcal{H}^{(d-1)}_x + \Delta_{\beta,\eta} \int_{\partial^*_g u \cap \mathcal{P}_d} d\mathcal{H}^{(d-1)}_x,$$

Set $\mathscr{W}^{+}_{\beta,\eta}(\cdot) = \mathscr{W}_{\beta,\eta}(\cdot \mid g_+)$ and, accordingly, $\mathscr{W}^{\pm}_{\beta,\eta}(\cdot) = \mathscr{W}_{\beta,\eta}(\cdot \mid g_\pm)$. As we shall see below this notation exactly corresponds to the color blind formalism of the FK-representation and to the construction of the measures $\mathbb{P}^{\beta,|\eta|}_{N,\pm}$ in (2.6). Thus, the Winterbottom functional (1.1) can be rewritten (recall that for $\eta \leqslant 0$, $\Delta_{\beta,\eta} = -\Delta_{\beta,|\eta|}$) as

$$\mathscr{W}_{\beta,\eta}(u) = \begin{cases} \mathscr{W}^{+}_{\beta,\eta}(u) & \text{for} \quad \eta \geqslant 0 \\ \mathscr{W}^{\pm}_{\beta,|\eta|}(u) - \Delta_{\beta,|\eta|} & \text{for} \quad \eta \leqslant 0 \end{cases} \tag{3.4}$$

In this way the functional $\mathscr{W}_{\beta,\eta}$ is unambiguously defined on $\mathrm{BV}(\mathrm{int}\, \widehat{\mathbb{D}}, \{\pm 1\})$ for every constant sign magnetic field η.

For $m \in]\hat{m}(\beta,\eta), m^*(\beta)[$ the Winterbottom shape $\mathscr{K}_{\beta,\eta}(v(m))$ fits into $\widehat{\mathbb{D}}$ and the function $\mathbb{1}_{\mathscr{K}_{\beta,\eta}(v(m))} \in \mathrm{BV}(\mathrm{int}\, \widehat{\mathbb{D}}, \{\pm 1\})$ is the stable minimizer of $\mathscr{W}_{\beta,\eta}$ in the following sense: For every $\delta > 0$ there exists $\epsilon > 0$ such that for every function $u \in \mathrm{BV}(\mathrm{int}\, \widehat{\mathbb{D}}, \{\pm 1\})$ with $\int_{\widehat{\mathbb{D}}} u(x)\, \mathrm{d}x \leqslant m/m^*_\beta$,

$$\mathscr{W}_{\beta,\eta}(u) \leqslant w^*_{\beta,\eta}(m) + \epsilon \Rightarrow \min_x \|u - \mathbb{1}_{x + \mathscr{K}_{\beta,\eta}(v(m))}\|_{\mathbb{L}_1(\widehat{\mathbb{D}})} \leqslant \delta. \tag{3.5}$$

Indeed, let $\tau_{\beta,\eta}$ be the support function of the Winterbottom shape $\mathscr{K}_{\beta,\eta}$ itself. Regardless of the sign of the monochromatic magnetic field η define the relaxed functional:

$$\widehat{\mathscr{W}}_{\beta,\eta}(u) = \int_{\partial^*_{g_+} u} \tau_{\beta,\eta}(\vec{n}_x)\, d\mathscr{H}^{d-1}_x.$$

Being the Wulff type functional (or, in an alternative terminology, being the mixed volume[39]) $\widehat{\mathscr{W}}_{\beta,\eta}$ is lower-semicontinuous with respect to the $\mathbb{L}_1(\widehat{\mathbb{D}})$-convergence. Furthermore, by the refinement[40] of the Brunn–Minkowski inequality,[39] the shifts and dilations of $\mathscr{K}_{\beta,\eta}$ are the only minimizers of $\widehat{\mathscr{W}}_{\beta,\eta}$ in the problems with the corresponding volume constraints. Hence, $\widehat{\mathscr{W}}_{\beta,\eta}$ satisfies the stability property (3.5). However, $\mathscr{W}_{\beta,\eta} \geqslant \widehat{\mathscr{W}}_{\beta,\eta}$ and their values coincide on the shifts and dilations of the Winterbottom shape $\mathscr{K}_{\beta,\eta}$. As a result, the stability property (3.5) is immediately inherited by the original functional $\mathscr{W}_{\beta,\eta}$.

In the partially wetting case $\Delta_{\beta,\eta} < \tau_\beta(\vec{e}_d)$ the minimum in (3.5) is over all admissible shifts $x \in \mathscr{P}_d$ along the substrate wall. In order to see this notice that in the latter situation the minimum $w^*_{\beta,\eta}(m)$ is strictly less than the unconstrained minimum of the Wulff functional $\mathscr{W}_\beta(u)$ over the functions $u \in \mathrm{BV}(\mathrm{int}\, \widehat{\mathbb{D}}, \{\pm 1\})$; $\int_{\widehat{\mathbb{D}}} u(x)\, \mathrm{d}x \leqslant m/m^*_\beta$.

Fix now a point $x \in \widehat{\mathbb{D}} \setminus \mathscr{P}_d$ and define $r = x_d > 0$. For any $\delta > 0$ and any function $u \in \mathrm{BV}(\mathrm{int}\, \widehat{\mathbb{D}}, \{\pm 1\})$ satisfying $\|u - \mathbb{1}_{x+\mathscr{K}_{\beta,\eta}(v(m))}\|_{\mathbb{L}_1(\widehat{\mathbb{D}})} \leqslant \delta$ one choose a section $r_\delta \leqslant r$ such that $\mathscr{H}^{d-1}(\{x_d = r_\delta\} \cap \{x : u(x) = -1\}) \leqslant \delta / r$. Modifying u as $u_\delta = +1$ on $x_d < r_\delta$ we obtain that $\mathscr{W}_\beta(u_\delta) \leqslant \mathscr{W}_{\beta,\eta}(u) + \delta \max \tau_\beta / r$ and, of course, that $\int_{\widehat{\mathbb{D}}} u_\delta(x)\, \mathrm{d}x \leqslant (m+\delta)/m^*_\beta$. As a result, a $\mathscr{W}_{\beta,\eta}$-minimizing sequence cannot converge to $\mathbb{1}_{x+\mathscr{K}_{\beta,\eta}}$.

3.2. Mesoscopic Phase Labels

In the $\mathbb{L}_1$-approach the local order parameter $\mathscr{M}_{N,K}$ is quantified by three values $\{-1, 0, 1\}$ according to the local proximity of the system to one of the pure phase. Such a renormalization procedure is delicate to implement directly. One way, proposed by Pisztora in ref. 3, relies on the FK representation of Ising model. A technical advantage to work with the FK representation, is the uniqueness of the measure in the thermodynamic limit even when there is breaking of symmetry for the Ising model. In a sense the FK measure is much less sensitive to boundary effects than the Gibbs measure, which underlines the required decoupling properties.

Typical Configurations for the FK Representation. As explained in (2.2), the FK counterpart of spontaneous magnetization is a uniform positive probability that the site 0 is connected to the boundary of arbitrarily large boxes with wired boundary conditions. The implementation of Pisztora's coarse graining requires an *a priori* stronger notion, namely percolation in slabs:

$$\textbf{(A2)} \quad \exists \delta > 0, \exists L_0 > 0, \forall L > L_0, \lim_{N \to \infty} \inf_{x,y \in S_{L,N}} \Phi^{\beta,f}_{S_{L,N}}(x \leftrightarrow y) > \delta,$$

where $S_{L,N}$ is the slab $\{|i_1| \leqslant L, -N \leqslant i_k \leqslant N, k = 2,..., d\}$. The importance of this notion was first realized in ref. 41 in the context of Bernoulli percolation. If **(A2)** is satisfied then there is percolation and $\beta > \beta_c$. The critical value $\hat{\beta}_c$ above which **(A2)** holds is called slab percolation threshold. We define $\mathfrak{B}$ as the subset of $]\hat{\beta}_c, \infty[$ for which assumption **(A1)** holds. The set $\mathfrak{B}$ differs from $]\hat{\beta}_c, \infty[$ by at most a countable number of points. In fact, it is conjectured that $\mathfrak{B} =]\beta_c, \infty[$.

Originally (see ref. 3) the estimates on typical configurations were devised in a nearest neighbor set-up. Inspection of the proof shows that the results extend readily to models with finite range interaction.

Property **(A2)** enables to obtain enhanced estimates on the percolation in domains by slicing these domains into slabs. Furthermore assumption **(A1)** implies a relaxation property of the system and therefore uniform estimates can be deduced for arbitrary boundary conditions.

We can now proceed to describe Pisztora's coarse graining. Let α be in $(0, 1)$ and fix $\zeta > 0$. On a coarse grained scale $K \in \mathbb{N}$, the typical configurations ω in the box $\mathbb{B}_K$ satisfy the 3 properties below:

1. There is a unique crossing cluster C^* in $\mathbb{B}_K$, i.e., a cluster which is connected to all the faces of the vertex boundary

$$\partial \mathbb{B}_K \triangleq \{x \in \mathbb{B}_K^c \mid \exists y \in \mathbb{B}_K, \text{ such that } J(x-y) > 0\}.$$

The faces are simply the faces of the box $\mathbb{B}_K$ enlarged by a factor depending on the range of the interaction.

2. Every open path in $\mathbb{B}_K$ with diameter larger than K^α is contained in C^*. (The diameter of a subset A of $\mathbb{Z}^d$ is $\sup_{x,\, y \in A} \|x-y\|_1$.)

3. The density of the crossing cluster in $\mathbb{B}_{K/2}$ is close to m^* with accuracy $\zeta > 0$

$$|C^*| \in [m^* - \zeta, m^* + \zeta] \frac{K^d}{2^d},$$

where $|\cdot|$ denotes the number of vertices in a set.

A configuration ω in $\mathbb{B}_K$ is good if it satisfies the 3 assertions above. It was proven in ref. 3 that for any $\beta \in \mathscr{B}$ and for large enough mesoscopic scales, good configurations are, uniformly over boundary conditions, typical in the following sense:

$$\inf_{\pi} \Phi_K^{\beta, \pi} (\omega \text{ is a good configuration in } \mathbb{B}_K) \geqslant 1 - \exp(-c_\zeta K^\alpha), \qquad (3.6)$$

where $c_\zeta > 0$ depends on the accuracy ζ of the coarse graining.

Mesoscopic Phase Labels. On the mesoscopic scale $K = 2^k$, each box $\widehat{\mathbb{B}}_{N,K}(x)$, centered at $x \in \widehat{\mathbb{D}}_{N,K}$ is labeled by the variable $u_{N,K}^{\zeta, \mathbf{FK}}(\omega, x)$

$$\forall x \in \widehat{\mathbb{D}}_{N,K}, \quad u_{N,K}^{\zeta, \mathbf{FK}}(\omega, x) \triangleq \begin{cases} 1 & \text{if } \omega \text{ is a good configuration in } \mathbb{B}_{2K}(Nx), \\ 0 & \text{otherwise,} \end{cases}$$

where the information recorded in $u_{N,K}^{\zeta, \mathbf{FK}}(\omega, x)$ takes into account the bond configuration in the larger box $\mathbb{B}_{2K}(Nx)$. We say that a block $\widehat{\mathbb{B}}_{N,K}(x)$ is regular if $u_{N,K}^{\zeta, \mathbf{FK}}(x) = 1$.

Estimate (3.6) is uniform over the boundary conditions, therefore it enables to control the variable $u_{N,K}^{\zeta, \mathbf{FK}}(x)$ independently of the events which occurs on boxes which are not $*$-connected to $\widehat{\mathbb{B}}_{N,K}(x)$. Following refs. 3

and 42 this implies stochastic domination of the field $\{x \in \widehat{\mathbb{D}}_{N,K} \mid u_{N,K}^{\zeta,\mathbf{FK}}(\omega, x) = 0\}$ by the Bernoulli site percolation measure with parameter $\exp(-c_\zeta K^\alpha)$ on $\widehat{\mathbb{D}}_{N,K}$; in particular,

$$\Phi_N^{\beta, |\eta|, \mathrm{w}}\left(u_{N,K}^{\zeta,\mathbf{FK}}(x_1) = 0,\ldots, u_{N,K}^{\zeta,\mathbf{FK}}(x_\ell) = 0\right) \leqslant \exp(-c_\zeta K^\alpha \ell). \tag{3.7}$$

Furthermore, the basic FK-estimate (3.6) enables to control the local magnetization profiles $\mathscr{M}_{N,K}$, for large enough mesoscopic scale $K = 2^k$, by a coarse graining which satisfies similar decoupling properties. Let x be in $\widehat{\mathbb{D}}_{N,K}$. Then for any $y \in \widehat{\mathbb{B}}_{N,K}(x)$, we define the mesoscopic phase label $u_{N,K}^\zeta$ as:

$$u_{N,K}^\zeta(\sigma, \omega, y) \triangleq \begin{cases} \operatorname{sign}(C^*) & \text{if} \quad u_{N,K}^{\zeta,\mathbf{FK}}(\omega, x) = 1 \quad \text{and} \\ & \quad |\mathscr{M}_{N,K}(\sigma, x) - \operatorname{sign}(C^*)\, m^*| < 2\zeta, \\ 0 & \text{otherwise}, \end{cases} \tag{3.8}$$

where C^* is the crossing cluster in $\mathbb{B}_K(x)$.

A block $\widehat{\mathbb{B}}_{N,K}(x)$ may have label 0 for two reasons: either $u_{N,K}^{\zeta,\mathbf{FK}}(\omega, x) = 0$, or ω is a good configuration on $\mathbb{B}_{2K}(Nx)$, but the local average magnetization $\mathscr{M}_{N,K}(x)$ on $\mathbb{B}_K(Nx)$ is non typical. The former case is covered by (3.7). In the latter case, the corresponding shift of the magnetization is entirely due to an abnormal coloring of the small (i.e., with the diameter less than K^α) ω-clusters intersecting $\mathbb{B}_K(Nx)$. The probability of such a deviation is, for each particular block $\mathbb{B}_K(Nx)$, bounded above by $\exp\{-c\zeta^2 K^{d(1-\alpha)}\}$. On the other hand, all small clusters intersecting $\mathbb{B}_K(Nx)$ lie inside $\mathbb{B}_{2K}(Nx)$, which already leads to the required decoupling properties. Notice that the coloring argument inside a good block is insensitive to the repainting of the $\mathfrak{g}$-cluster into -1 on the event $\mathfrak{I}_N$. We refer to the original article[3] for the precise workout of these estimates and to ref. 9 where the proof has been adapted to the setup we employ here.

As a result we obtain that there there exists a sequence $\{\rho_K^\zeta\}$ with $\lim_{K\to\infty} \rho_K^\zeta = 0$, such that the field $\{x \in \widehat{\mathbb{D}}_{N,K} \mid u_{N,K}^\zeta(\sigma, \omega, x) = 0\}$ is stochastically dominated by the Bernoulli site percolation measure $\mathbb{P}_{\text{perc}}^{\rho_K^\zeta}$ on $\widehat{\mathbb{D}}_{N,K}$. In particular, for $\eta \geqslant 0$

$$\mathbb{P}_N^{\beta,\eta}(u_{N,K}^\zeta(x_1) = 0,\ldots, u_{N,K}^\zeta(x_\ell) = 0) \leqslant (\rho_K^\zeta)^\ell. \tag{3.9}$$

Remark 3.1. If for two different points $x, y \in \widehat{\mathbb{D}}_{N,K}$ the corresponding $u_{N,K}^\zeta$-phase labels have opposite signs, that is if $u_{N,K}^\zeta(x)\, u_{N,K}^\zeta(y) = -1$, then on any finer scale $K' \leqslant K$ any $*$-connected chain of $\widehat{\mathbb{B}}_{N,K'}$ blocks joining

$\widehat{\mathbb{B}}_{N,K}(x)$ to $\widehat{\mathbb{B}}_{N,K}(y)$ contains at least one block with zero $u_{N,K'}^{\zeta,\mathbf{FK}}$ (and, hence, with zero $u_{N,K'}^{\zeta}$)-label. This follows from the fact that 2 boxes with opposite labels cannot have a common crossing FK-cluster; thus they are necessarily separated by a contour of 0-blocks.

Remark 3.2. The modification of the boundary interaction has no impact on the coarse graining. Since the slab percolation threshold hypothesis requires percolation on slab with free boundary condition, the presence of a positive boundary magnetic field can only improve the estimates in view of FKG monotonicity. Furthermore, in each box touching the wall, the event that a crossing cluster is connected to the ghost point $\mathfrak{g}$ occurs with overwhelming probability (the slab percolation hypothesis **(A2)** is valid uniformly over the boundary conditions). Finally the control of the density of the crossing cluster (property 3 in the definition of a good block) is unchanged because the boundary field has no influence on the bulk properties due to the hypothesis **(A1)**.

The mesoscopic phase labels $u_{N,K}^{\zeta}$ are $\{\pm 1, 0\}$-valued function on $\widehat{\mathbb{D}}$ which are piece-wise constant with respect to the partition (1.12). We are now going to prove that the $\mathbb{L}_1$-difference between the local magnetization profiles $\mathscr{M}_{N,K}$ and the phase labels $u_{N,K}^{\zeta}$ vanishes on the exponential scale. Thus, from the point of view of the $\mathbb{L}_1$-theory these objects are indistinguishable, and one is entitled to switch freely from one to the other. In particular, the fundamental exponential tightness result which we shall establish below in Subsection 3.4 comes naturally in the context of the $\{\pm 1, 0\}$-valued phase labels.

3.3. Relation to Magnetization Profiles

Lemma 3.1. Let a positive boundary magnetic field $\eta \geqslant 0$ and a number $\nu < 1/d$ be fixed. For any $\delta > 0$, one can choose the accuracy ζ of the coarse graining and a finite scale $K_0 = K_0(\delta, \beta)$, such that

$$\frac{1}{N^{d-1}} \max_{K_0 \leqslant K \leqslant N^{\nu}} \log \mathbb{P}_N^{\beta,\eta}(\|\mathscr{M}_{N,K} - m^* u_{N,K}^{\zeta}\|_{\mathbb{L}_1} > \delta) \leqslant -c(\zeta, K)\, N^{1-d\nu}. \quad (3.10)$$

Remark 3.3. By (2.7) and (2.6) the super surface order exponential estimate (3.10) holds, in the case of negative boundary magnetic field $\eta \leqslant 0$, also for the conditional measure $\mathbb{P}_{N,\pm}^{\beta,|\eta|}$.

Proof. Recall that the coarse graining is initially defined in terms of the mesoscopic FK-variables $u_{N,K}^{\zeta,\mathbf{FK}}(\omega)$ which depend only on the bond configurations. The phase label $u_{N,K}^{\zeta}(\sigma,\omega)$ takes also into account the random coloring of good FK-blocks. In fact,

$$\|\mathscr{M}_{N,K} - m^* u_{N,K}^{\zeta}\|_1 \leqslant \zeta + \frac{2}{|\widehat{\mathbb{D}}_{N,K}|} \sum_{x \in \widehat{\mathbb{D}}_{N,K}} 1_{u_{N,K}^{\zeta}(x)=0},$$

and the claim of the lemma follows by the domination by Bernoulli site percolation (3.9), actually for any choice of $\zeta < \delta$. ∎

3.4. Tightness Theorem for Mesoscopic Phase Labels

We shall formulate a tightness result which holds under fairly general phase boundary conditions outside $\widehat{\mathbb{D}}$. The proof is a straightforward adaptation of the argument used to verify the claim of the tightness Theorem 2.2.1 in ref. 12 and, as we shall briefly indicate in Remark 3.4 below, it can be trivially extended to cover the case of q-valued phase labels as, for example, recently considered in ref. 11.

Given $N = 2^n$ and $K = 2^k$ $(n \geqslant k)$ consider, as in (1.12), the following splitting of $\mathbb{R}^d$:

$$\mathbb{R}^d = \bigvee_{x \in \mathbb{Z}_{N,K}^d} \widehat{\mathbb{B}}_{N,K}(x), \tag{3.11}$$

In what follows we shall use the same notation as in Subsection 3.1. Thus, $\mathscr{O}$ is a fixed smooth neighborhood of $\widehat{\mathbb{D}}$. A boundary condition is an element $g \in \mathrm{BV}(\mathscr{O} \setminus \widehat{\mathbb{D}}, \{\pm 1\})$, which is piece-wise constant with respect to the partition (3.11) on some (M, L)-scale. Notice that such boundary condition g is automatically piece-wise constant on all further (N, K) scales with $N \geqslant M$ and $N/K \geqslant M/L$.

A family $\{u_{N,K}\}$ of mesoscopic phase labels is said to be compatible with a boundary condition g; $u_{N,K} \sim g$, if g is piece-wise constant on the (N, K)-scale and the following phase rigidity condition **(R)** is satisfied:

(R) Define $w_{N,K} = u_{N,K} \vee g$ as in (3.1). Then for every $x \in \mathbb{D}_{N,K}$ and $y \in \mathbb{Z}_{N,K}^d$ with $w_{N,K}(x)\, w_{N,K}(y) = -1$ and for every $L \leqslant K$ any $*$-connected chain of $\widehat{\mathbb{B}}_{N,L}$-blocks leading from $\widehat{\mathbb{B}}_{N,K}(x)$ to $\widehat{\mathbb{B}}_{N,K}(y)$ necessarily contains at least one $\widehat{\mathbb{B}}_{N,L}$-block with the $u_{N,L}$-label zero.

In order to state the tightness result we need to define the perimeter of a BV-function $u \in \mathrm{BV}(\mathrm{int}\, \widehat{\mathbb{D}}, \{\pm 1\})$ under a boundary condition g:

$$\mathscr{P}_g(u) = \mathscr{H}_{d-1}(\partial_g^* u).$$

It is well known[38] that for every $a \geqslant 0$ and for every boundary condition g the set

$$\mathscr{C}_g(a) = \{u \in \mathrm{BV}(\mathrm{int}\ \hat{\mathbb{D}}, \{\pm 1\}) \mid \mathscr{P}_g(u) \leqslant a\}$$

is compact in $\mathbb{L}_1(\hat{\mathbb{D}})$. Let P_g be the minimal perimeter of the functions compatible with the boundary condition g.

Before deriving an exponential tightness for the mesoscopic phase labels under measures with boundary fields, we first state a general result for an abstract measure $\mathbb{P}$ and general boundary conditions

Theorem 3.1. Let a sequence of non-negative numbers $\{\rho_K\}$ satisfy $\lim_{K\to\infty} \rho_K = 0$. Assume that a family of random mesoscopic phase labels $\{u_{N,K}\}$ on $\hat{\mathbb{D}}$ satisfies the following decoupling condition **(D)**:

(D) For every N and K the zero-label field $\{x \in \mathbb{Z}^d_{N,K} \mid u_{N,K}(x) = 0\}$ under the measure $\mathbb{P}$ is stochastically dominated by the Bernoulli site percolation measure $\mathbb{P}^{\rho_K}_{\mathrm{perc}}$.

Fix a number $v < 1/d$. Then for every $\delta > 0$ and for each boundary condition g there exists a finite scale $K_0 = K_0(d, \delta)$ and a constant $c = c(d) > 0$, such that, uniformly in $a > P_g$,

$$\limsup_{N\to\infty} \max_{K_0 \leqslant K \leqslant N^v} \frac{1}{N^{d-1}} \log \mathbb{P}[u_{N,K} \sim g\,;\, u_{N,K} \notin \mathscr{V}(\mathscr{C}_g(a), \delta)] \leqslant -\frac{c}{K_0^{d-1}}\, a,$$

where $\mathscr{V}(\mathscr{C}_g(a), \delta)$ denotes the δ-neighborhood in $\mathbb{L}_1$ of $\mathscr{C}_g(a)$.

Remark 3.4. The tightness result above could be trivially extended to the case of $\{0, 1,..., q\}$-valued phase labels. One needs only to modify the rigidity condition **(R)** (to require a zero block in any $*$-connected chain between two blocks with different non-zero labels), and redefine $\mathscr{P}_g(u)$ as the $\mathscr{H}^{(d-1)}$-measure of the jump set of $u \vee g$.[43] If $u_{N,K}$ is a $\{0, 1..., q\}$-valued phase label and g is a boundary condition which satisfies such modified assumption **(R)** and the decoupling assumption **(D)**, then, for every $i = 1,..., q$, the $\{0, \pm 1\}$-label $u^i_{N,K}$ defined as

$$u^i_{N,K}(x) = \begin{cases} 1 & \text{if } \ u_{N,K}(x) = i \\ 0 & \text{if } \ u_{N,K}(x) = 0 \\ -1 & \text{otherwise} \end{cases}$$

and the boundary condition g^i similarly defined already satisfies the assumptions of Theorem 3.1. It remains to notice that

$$\{u_{N,K} \in \mathscr{V}(\mathscr{C}_g(a), \delta)\} \supset \bigcap_{i=1}^{q} \{u^i_{N,K} \in \mathscr{V}(\mathscr{C}_{g^i}(a/q), \delta/q)\}.$$

Proof. The proof essentially follows Theorem 2.2.1 in ref. 12. The only difference with the periodic case discussed there is the existence of open contours attached to the boundary. But, for any choice of the macroscopic boundary condition g (as defined above), these boundary contours are, in the language of ref. 12, large, that is their diameter exceeds $C_d \log N$ on all sufficiently big (N, K) renormalization scales. Let us briefly recall the three main steps of the argument:

The first step is to check that the volume of 0-blocks is negligible. This simply follows by the assumption **(D)**:

$$\mathbb{P}\left(\#\{x \in \hat{\mathbb{D}}_{N,K} : u_{N,K}(x) = 0\} \geqslant \delta \left(\frac{N}{K}\right)^d\right) \leqslant \exp\left(-\delta \left(\frac{N}{K}\right)^d \log\left(\frac{\delta}{\rho_K}\right)\right). \tag{3.12}$$

The second and the third steps are devoted to the control of the regions surrounded by contours, i.e., by connected surfaces of 0-blocks. By phase rigidity assumption **(R)** the contours play the role of mesoscopic interfaces separating regions with mesoscopic phase labels of different signs so that they contribute to the total perimeter of the configurations $u_{N,K}$. We distinguish between two types of contours, the small contours with diameter smaller than $C_d \log N$ (where C_d is a given constant) and the remaining contours, namely the large ones.

Peierls type estimates can be applied to bound the probability of events such that the collection $\{\Gamma_i\}$ of large contours has a total area larger than $a(\frac{N}{K})^{d-1}$

$$\mathbb{P}\left(\sum_{\Gamma_i \text{ large}} |\Gamma_i| \geqslant \left(\frac{N}{K}\right)^{d-1} a\right) \leqslant \exp\left(-c_1 a \left(\frac{N}{K}\right)^{d-1}\right). \tag{3.13}$$

The small contours are not controlled in terms of their total area but of the volume of the regions they surround. By the choice of the macroscopic boundary condition g there are no open small contours, and, proceeding exactly as in the proof of Theorem 2.2.1 in ref. 12, one can show that

$$\mathbb{P}\left(\sum_{\Gamma_i \text{ small}} \text{vol}(\Gamma_i) \geqslant \delta \frac{N^d}{K^d}\right) \leqslant \exp\left(-c_2 \frac{\delta N^d}{K^d \log N}\right). \tag{3.14}$$

As a consequence of the previous estimate, the small contours will have no contribution to the $\mathbb{L}_1$-norm. Combining estimates (3.12), (3.13) and (3.14), we arrive to the claim of the Theorem. ▮

Remark 3.5. The last step of the proof should be modified for general, that is not necessarily piece-wise constant with respect to the partition (3.11), boundary conditions $g \in \mathrm{BV}(\mathcal{O} \setminus \widehat{\mathbb{D}}, \{\pm 1\})$. See also refs. 10 and 11 where small contours have been ignored and, accordingly tightness estimates have been derived only on diverging (N, K)-scales.

Let us go back to the phase labels $\{u^{\zeta}_{N,K}\}$ we consider here. The boundary condition g is given by $g = g^{+} \equiv 1$ in the case of positive boundary magnetic fields $\eta \geqslant 0$ and by $g = g^{\pm}$ specified in (3.3) in the case of negative η.

Proposition 3.1. Let $\beta \in \mathfrak{B}$. Then there exists $C(\beta, d) > 0$ such that for all δ positive, one can find $K_0 = K_0(\delta, \beta, d)$, $\zeta = \zeta(\delta)$ such that

1. In the case of positive magnetic fields $\eta \geqslant 0$,

$$\limsup_{N \to \infty} \frac{1}{N^{d-1}} \max_{K_0 \leqslant K \leqslant N^{\nu}} \log \mathbb{P}^{\beta,\eta}_{N}\left[u^{\zeta}_{N,K} \notin \mathcal{V}(\mathscr{C}_{g^+}(a), \delta\right] \leqslant -\frac{C}{K_0^{d-1}}\, a.$$

2. In the case of negative magnetic fields $\eta < 0$,

$$\limsup_{N \to \infty} \frac{1}{N^{d-1}} \max_{K_0 \leqslant K \leqslant N^{\nu}} \log \mathbb{P}^{\beta,|\eta|}_{N,\pm}\left[u^{\zeta}_{N,K} \notin \mathcal{V}(\mathscr{C}_{g^{\pm}}(a), \delta)\right] \leqslant -\frac{C}{K_0^{d-1}}\, a + |\Delta_{\beta,\eta}|.$$

Proof. Let us briefly work out the $\eta < 0$ case: Recall that $\mathbb{P}^{\beta\,|\eta|}_{N,\pm}(\cdot) = \mathbb{P}^{\beta,|\eta|}_{N}(\cdot\,; \mathfrak{I}_N)/\mathbb{P}^{\beta\,|\eta|}_{N}(\mathfrak{I}_N)$ On the other hand, any phase label $u^{\zeta}_{N,K}$ which is compatible with $\mathfrak{I}_N$ is, after the repainting of the g-cluster into -1, automatically compatible with the boundary condition $g^{\pm}$ and, therefore, satisfies the conditions of Theorem 3.1 (under the measure $\mathbb{P}^{\beta,|\eta|}_{N}$). Thus, the claim of the proposition follows from the general tightness Theorem 3.1 and, in the case of negative magnetic fields $\eta < 0$, by the representation formula for the boundary surface tension (2.7). ▮

4. THE PROOFS

We going are to derive sharp asymptotics involving the surface tension. As mentioned in Subsection 3.4, the case of negative boundary field requires more care, therefore we focus on this case in the following.

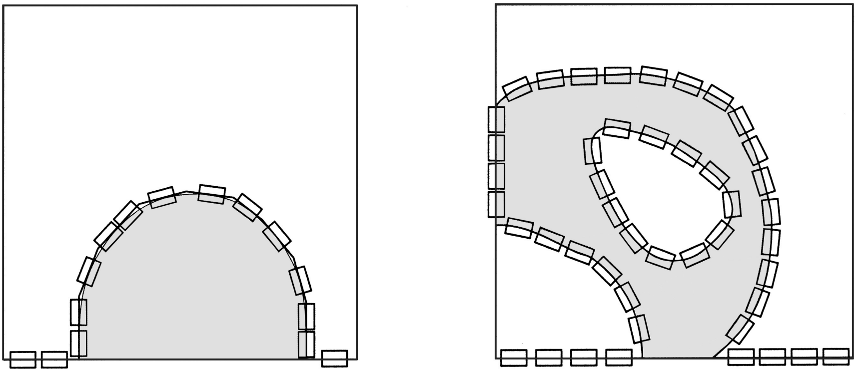

Fig. 2. Left: The approximate Winterbottom shape $\mathscr{K}^{\delta}_{\beta,\eta}(v(m))$ and the boxes of Proposition 4.1. Right: A function $v \in \mathrm{BV}(\mathrm{int}\,\hat{\mathbb{D}}, \{\pm 1\})$ (the shaded area corresponds to $v \equiv -1$), and the boxes of Theorem 4.1.

4.1. Approximation Result I

Given a boundary magnetic field η and $m \in]\bar{m}_{\beta,\eta}, m^*_\beta[$, we are going to approximate the Winterbottom shape $\mathscr{K}_{\beta,\eta}(v(m))$ by regular sets (see Fig. 2).

Lemma 4.1. For any $\delta > 0$ one can construct a polyhedral set $\mathscr{K}^{\delta}_{\beta,\eta}$ satisfying:

$$\|\mathbb{1}_{\mathscr{K}^{\delta}_{\beta,\eta}} - \mathbb{1}_{\mathscr{K}_{\beta,\eta}}\|_{\mathbb{L}_1} \leqslant \delta \qquad \text{and} \qquad |\mathscr{W}_{\beta,\eta}(\mathscr{K}^{\delta}_{\beta,\eta}) - \mathscr{W}_{\beta,\eta}(\mathscr{K}_{\beta,\eta})| \leqslant \delta.$$

Proof. It is a standard result of the theory of convex bodies (see, e.g., ref. 39, Theorem 1.8.13) that for every $\delta > 0$ there exists a convex polyhedral set $\mathscr{K}^{\delta}_{\beta}$ such that the Hausdorff distance $\mathrm{d}_{\mathrm{Hausd}}(\mathscr{K}^{\delta}_{\beta}, \mathscr{K}_\beta) \leqslant \delta$. Of course, the support function τ^{δ}_{β} of such $\mathscr{K}^{\delta}_{\beta}$ satisfies

$$\max_{\vec{n} \in \mathbb{S}^{d-1}} |\tau^{\delta}_{\beta}(\vec{n}) - \tau_\beta(\vec{n})| \leqslant \delta. \tag{4.1}$$

Define now

$$\mathscr{K}^{\delta}_{\beta,\eta} = \mathscr{K}^{\delta}_{\beta} \cap \{x : x_d \geqslant -\Delta_{\beta,\eta}\}.$$

A comparison with (1.2) reveals that

$$\mathrm{d}_{\mathrm{Hausd}}(\mathscr{K}^{\delta}_{\beta,\eta}, \mathscr{K}_{\beta,\eta}) \leqslant \delta \qquad \text{and} \qquad \max_{\vec{n} \in \mathbb{S}^{d-1}} |\tau^{\delta}_{\beta,\eta}(\vec{n}) - \tau_{\beta,\eta}(\vec{n})| \leqslant \delta, \tag{4.2}$$

where $\tau^{\delta}_{\beta,\eta}$ and $\tau_{\beta,\eta}$ are the support functions of $\mathscr{K}^{\delta}_{\beta,\eta}$ and $\mathscr{K}_{\beta,\eta}$ respectively. Now, for every bounded convex $\mathscr{K}\subset\mathbb{R}^d$ the volume $\mathrm{vol}_d(\mathscr{K})$ can be written in terms of its support function $\tau_{\mathscr{K}}$ as

$$\mathrm{vol}_d(\mathscr{K})=\frac{1}{d}\int_{\partial\mathscr{K}}\tau_{\mathscr{K}}(\vec{n}_x)\,\mathrm{d}\mathscr{H}^{(d-1)}_x.$$

Consequently, (4.2) implies that

$$\left|\int_{\partial\mathscr{K}^{\delta}_{\beta,\eta}}\tau^{\delta}_{\beta,\eta}(\vec{n}_x)\,\mathrm{d}\mathscr{H}^{(d-1)}_x-\int_{\partial\mathscr{K}_{\beta,\eta}}\tau_{\beta,\eta}(\vec{n}_x)\,\mathrm{d}\mathscr{H}^{(d-1)}_x\right|\leqslant d\delta\,|\partial\mathscr{K}_{\beta,\eta}|.$$

As we have already mentioned in the end of Subsection 3.1,

$$\int_{\partial\mathscr{K}_{\beta,\eta}}\tau_{\beta,\eta}(\vec{n}_x)\,\mathrm{d}\mathscr{H}^{(d-1)}_x=\mathscr{W}_{\beta,\eta}(\mathscr{K}_{\beta,\eta}).$$

On the other hand, for $\vec{n}=-\vec{e}_d$, the support function $\tau^{\delta}_{\beta,\eta}$ equals to $\Delta_{\beta,\eta}$. Moreover, for every $x\in\partial\mathscr{K}^{\delta}_{\beta,\eta}\setminus\{y: y_d=-\Delta_{\beta,\eta}\}$ any supporting hyperplane to $\partial\mathscr{K}^{\delta}_{\beta,\eta}$ at x is also a supporting hyperplane to $\partial\mathscr{K}^{\delta}_{\beta}$. It follows that the support function $\tau^{\delta}_{\beta,\eta}(\vec{n}_x)=\tau^{\delta}_{\beta}(\vec{n}_x)$ on $x\in\partial\mathscr{K}^{\delta}_{\beta,\eta}\setminus\{y: y_d=-\Delta_{\beta,\eta}\}$. Thus, as it follows now from (4.1),

$$\left|\int_{\partial\mathscr{K}^{\delta}_{\beta,\eta}}\tau^{\delta}_{\beta,\eta}(\vec{n}_x)\,\mathrm{d}\mathscr{H}^{(d-1)}_x-\mathscr{W}_{\beta,\eta}(\mathscr{K}^{\delta}_{\beta,\eta})\right|\leqslant\delta\,|\partial\mathscr{K}^{\delta}_{\beta,\eta}|.$$

Since by (4.2), $|\mathrm{vol}_d(\mathscr{K}^{\delta}_{\beta,\eta})-\mathrm{vol}_d(\mathscr{K}_{\beta,\eta})|\leqslant\delta\,|\partial\mathscr{K}_{\beta,\eta}|$, and both $|\partial\mathscr{K}_{\beta,\eta}|$ and $|\partial\mathscr{K}^{\delta}_{\beta,\eta}|$ are bounded above by some finite constant $c_1(\beta,d)<\infty$, we finally obtain the following estimate

$$|\mathscr{W}_{\beta,\eta}(\mathscr{K}^{\delta}_{\beta,\eta})-\mathscr{W}_{\beta,\eta}(\mathscr{K}_{\beta,\eta})|\leqslant c_2(\beta,d)\,\delta,$$

and the claim of the theorem follows once we redefine $\mathscr{K}^{\delta}_{\beta,\eta}=\mathscr{K}^{\delta/(c_2\vee 1)}_{\beta,\eta}$. ∎

4.2. The Lower Bound

As usual, we fix a number $0<\nu<1/d$.

Proposition 4.1. Assume that $\beta\in\mathfrak{B}$ and all the components of the boundary magnetic field $\eta\in\mathbb{R}^r$ have the same sign. Assume, furthermore, that $\Delta_{\beta,\eta} > -\tau_\beta(\vec{e}_d)$. For every $\delta > 0$, there is a finite scale $K_0(\delta)$ such that for every $m \in]\bar{m}(\beta,\eta), m^*(\beta)[$,

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \min_{K_0(\delta)\leqslant K\leqslant N^\nu} \log \mu_{N,+}^{\beta,\eta}(\|\mathscr{M}_{N,K} - m^* \mathbb{1}_{\mathscr{K}_{\beta,\eta}(v(m))}\|_{\mathbb{L}_1} \leqslant \delta)$$
$$\geqslant -w^\star_{\beta,\eta}(m) - o(\delta),$$

where minimal surface energy value $w^\star_{\beta,\eta}(m)$ has been defined in (1.11), and the function $o(\cdot)$ depends only on β and η and vanishes as the resolution δ goes to zero.

Proof. We shall give the proof only in the more difficult case of negative boundary fields $\eta < 0$. Also according to Lemma 3.1, it will be sufficient to derive the proposition with $\mathscr{M}_{N,K}$ replaced by $m^* u^\zeta_{N,K}$ (for ζ small enough).

Starting from the approximate shapes $\mathscr{K}^\delta_{\beta,\eta}$ let us use the transformation (1.3) to define the scaled polyhedral approximation $\mathscr{K}^\delta_{\beta,\eta}(v(m))$ of $\mathscr{K}_{\beta,\eta}(v(m))$. By the approximation result of Lemma 4.1 it will be enough to prove Proposition 4.1 with $\mathscr{K}^\delta_{\beta,\eta}(v(m))$ instead of $\mathscr{K}_{\beta,\eta}(v(m))$.

Define $\mathscr{P}_d^{\hat{\mathbb{D}}} = \mathscr{P}_d \cap \hat{\mathbb{D}}$. Since $\mathscr{K}^\delta_{\beta,\eta}(v(m))$ is polyhedral and convex it is an easy matter to show that there exists a side length $h > 0$ and a finite number ℓ of disjoint parallelepipeds $\hat{R}^1,\dots,\hat{R}^\ell$ with bases $\hat{B}^1,\dots,\hat{B}^\ell$ included in $\partial^\pm\mathscr{K}^\delta_{\beta,\eta} \equiv \partial\mathscr{K}^\delta_{\beta,\eta} \,\triangle\, \mathscr{P}_d^{\hat{\mathbb{D}}}$ of side length h and height δh such that:

(Ia) The sets $\hat{B}^1,\dots,\hat{B}^\ell$ cover $\partial^\pm\mathscr{K}^\delta_{\beta,\eta}$ up to a set of measure less than δ denoted by $\hat{U}^\delta = \partial^\pm\mathscr{K}^\delta_{\beta,\eta} \setminus \bigcup_{i=1}^\ell \hat{B}^i$ and they satisfy

$$\left| \sum_{\vec{n}_i \neq -\vec{e}_d} \tau_\beta(\vec{n}_i)\,|\hat{B}^i| + \Delta_{\beta,|\eta|} \sum_{\vec{n}_i = -\vec{e}_d} |\hat{B}^i| - \mathscr{W}^\pm_{\beta,|\eta|}(\mathscr{K}^\delta_{\beta,\eta}) \right| \leqslant \delta, \tag{4.3}$$

where $\vec{n}_i$ is the unit normal to $\hat{B}^i$ (and, hence, to the corresponding facet of $\partial^\pm\mathscr{K}^\delta_{\beta,\eta}$).

(Ib) For any $i = 1,\dots,\ell$ the base $\hat{B}^i$ divides $\hat{R}^i$ into two parallelepipeds $\hat{R}^{i,+}$ and $\hat{R}^{i,-}$, such that in the case of $\vec{n}_i \neq -\vec{e}_d$, $\hat{R}^{i,-} \subseteq \mathscr{K}^\delta_{\beta,\eta}$ and $\hat{R}^{i,+} \subseteq \hat{\mathbb{D}} \setminus \mathscr{K}^\delta_{\beta,\eta}$, whereas in the case of $\vec{n}_i = -\vec{e}_d$ the corresponding base $\hat{B}^i \subset \mathscr{P}_d^{\hat{\mathbb{D}}}$, while $\hat{R}^{i,+} \subseteq \hat{\mathbb{D}} \setminus \mathscr{K}^\delta_{\beta,\eta}$ and $\hat{R}^{i,-} \subset \{x : x_d \leqslant 0\}$.

In order to enforce a microscopic interface close to the polyhedral set $K^\delta_{\beta,\eta}$, we define

$$\mathscr{A} = \bigcap_{i=1}^{\ell} \{\partial^- R^i_N \nleftrightarrow \partial^+ R^i_N\}. \tag{4.4}$$

Let us also introduce the set $\widetilde{\mathscr{A}}$ chosen such that for any configuration ξ in $\mathscr{A} \cap \widetilde{\mathscr{A}}$ the bonds inside and outside $N\mathscr{K}^{\delta}_{\beta,\eta}$ are decoupled. For this, it is enough to close the bonds which are in a neighborhood of $U^{\delta}_N = N\widehat{U}^{\delta}$ (the microscopic counterpart of $\widehat{U}^{\delta}$)

$$\widetilde{\mathscr{A}} = \{\xi \in \Xi_{N,\mathbf{J},\eta},\ \xi_b = 0, \text{ if } \operatorname{dist}(b, U^{\delta}_N) \leqslant 10\}.$$

We get

$$\begin{aligned} &\mathbb{P}^{\beta,|\eta|}_{N,\pm}(\|u^{\zeta}_{N,K} - \mathbb{1}_{\mathscr{K}_{\beta,\eta}}\|_{\mathbb{L}_1} \leqslant \delta) \\ &\qquad \geqslant \mathbb{P}^{\beta,|\eta|}_{N,\pm}\left(\|u^{\zeta}_{N,K} - \mathbb{1}_{\mathscr{K}^{\delta}_{\beta,\eta}}\|_{\mathbb{L}_1} \leqslant \frac{\delta}{2} \,\middle|\, \mathscr{A} \cap \widetilde{\mathscr{A}}\right) \mathbb{P}^{\beta,|\eta|}_{N,\pm}(\mathscr{A} \cap \widetilde{\mathscr{A}}). \end{aligned} \tag{4.5}$$

We first check that

$$\liminf_{N\to\infty} \min_{K_0(\delta)\leqslant K\leqslant N^{\nu}} \mathbb{P}^{\beta,|\eta|}_{N,\pm}\left(\|u^{\zeta}_{N,K} - \mathbb{1}_{\mathscr{K}^{\delta}_{\beta,\eta}}\|_{\mathbb{L}_1} \leqslant \frac{\delta}{2} \,\middle|\, \mathscr{A} \cap \widetilde{\mathscr{A}}\right) \geqslant \frac{1}{4}. \tag{4.6}$$

From the argument developed in the proof of Proposition 3.1, we are going to check that in each of the two connected components of $\widehat{\mathbb{D}} \setminus \mathscr{K}^{\delta}_{\beta,\eta}$ the phase labels $u^{\zeta}_{N,K}$ are, with a uniformly positive probability, close to the equilibrium values (for the $\mathbb{L}_1$-norm).

Since $\mathscr{A} \cap \widetilde{\mathscr{A}}$ decouples the connected components, it is enough to consider the mesoscopic phase labels in the interior of the regions decoupled by $\mathscr{A} \cap \widetilde{\mathscr{A}}$. As the boundary of $\partial^{\pm}\mathscr{K}^{\delta}_{\beta,\eta}$ is regular, these regions are almost exhausted by the mesoscopic boxes for any scale $K \in [K_0, N^{\nu}]$. Using the terminology of Proposition 3.1, we assert that with large probability, the mesoscopic label configurations are almost uniformly constant and contain only contours which have no contribution in terms of the $\mathbb{L}_1$-norm. According to estimate (3.14), the volume of the regions surrounded by the small contours is arbitrarily small in the thermodynamic limit. Furthermore, from the usual Peierls estimates, the volume of the large contours is negligible as well. This implies (4.6).

Finally as $\mathscr{A} \cap \widetilde{\mathscr{A}} \subset \mathfrak{I}$, we see by applying (4.6) that

$$\mathbb{P}^{\beta,|\eta|}_{N,\pm}(\|u^{\zeta}_{N,K} - \mathbb{1}_{\mathscr{K}_{\beta,\eta}}\|_{\mathbb{L}_1} \leqslant \delta) \geqslant \frac{1}{4}\,\mathbb{P}^{\beta,|\eta|}_{N,\pm}(\mathscr{A} \cap \widetilde{\mathscr{A}}) = \frac{1}{4}\,\frac{\Phi^{\beta,|\eta|,\mathrm{w}}_N(\mathscr{A} \cap \widetilde{\mathscr{A}})}{\Phi^{\beta,|\eta|,\mathrm{w}}_N(\mathfrak{I})}. \tag{4.7}$$

The denominator can be easily estimated by (2.7)

$$\Phi^{\beta,|\eta|,\mathrm{w}}_N(\mathfrak{I}) = \exp(-\Delta_{\beta,|\eta|} N^{d-1} + o(N^{d-1})). \tag{4.8}$$

On the other hand, as we are trying to bound probabilities from below, even under the conditioning on $\widetilde{\mathscr{A}}$ one can still apply FKG arguments (non-crossing is a decreasing event) and use (2.3) and (2.7) to decouple between different $\hat{B}_i$-events which constitute $\mathscr{A}$ in (4.4). Therefore, since, $\Phi_N^{\beta,|\eta|,\mathrm{w}}(\widetilde{\mathscr{A}}) \geqslant e^{-c_3 \delta N^{d-1}}$, we are entitled to conclude that

$$\Phi_N^{\beta,|\eta|,\mathrm{w}}(\mathscr{A} \cap \widetilde{\mathscr{A}}) \geqslant \exp\left\{ -N^{d-1} \left(\sum_{\vec{n}_i \neq -\vec{e}_d} \tau_\beta(\vec{n}_i) \, |\hat{B}^i| + \Delta_{\beta,|\eta|} \sum_{\vec{n}_i = -\vec{e}_d} |\hat{B}^i| \right) - (c_3\delta + o(\delta)) \, N^{d-1} \right\}. \tag{4.9}$$

Combining (4.9) with the estimates (4.7) and (4.8); the approximation property (4.3); the choice of the polyhedral set $\mathscr{K}^{\delta}_{\beta,\eta}(v(m))$ through Lemma 4.1 and the representation (3.4) of $\mathscr{W}^{\pm}_{\beta,\eta}$, we arrive to the claim of Proposition 4.1. ▮

4.3. Approximation Result II

Approximation results used in the proof of the upper bound should apply to any $v \in \mathrm{BV}(\mathrm{int}\, \hat{\mathbb{D}}, \{\pm 1\})$. On the other hand, on the level of the precision provided by the $\mathbb{L}_1$-theory the underlying probabilistic estimates are less delicate than those needed for the proof of the lower bound for the distinguished almost optimal polyhedral shapes of the preceding subsection. In fact, global bulk relaxation properties play no rôle at all, and the $\mathbb{L}_1$-upper bound is just a coarse estimate on the surface tension price of the localized interface. In particular, the approximation construction of ref. 9 suffices and the proof of the corresponding result in the latter paper literally goes through. As in the case of the lower bound let us concentrate on the more difficult case of the negative boundary magnetic field $\eta \leqslant 0$. We claim:

Theorem 4.1. Let $v \in \mathrm{BV}(\mathrm{int}\, \hat{\mathbb{D}}, \{\pm 1\})$. For any δ positive, there exists h positive and a finite number ℓ of disjoint parallelepipeds $\hat{R}^1,..., \hat{R}^\ell$ with basis $\hat{B}^1,..., \hat{B}^\ell$ of size h and height δh such that:

(IIa) For every $i = 1,..., \ell$ the base $\hat{B}^i$ divides $\hat{R}^i$ into two parallelepipeds $\hat{R}^{i,+}$ and $\hat{R}^{i,-}$ and we denote by $\vec{n}_i$ the normal to $\hat{B}^i$. Either $\hat{B}^i \cap \partial\hat{\mathbb{D}} = \varnothing$ and $\hat{R}^i$ is included in $\hat{\mathbb{D}}$, or $\hat{B}^i$ is included in $\partial\hat{\mathbb{D}} \setminus \mathscr{P}_d$ and only $\hat{R}^{i,-}$ is included in $\hat{\mathbb{D}}$, or $\hat{B}^i$ is included in $\mathscr{P}_d$ and only $\hat{R}^{i,+}$ is included in $\hat{\mathbb{D}}$.

(IIb) The parallelepipeds $\hat{R}^1,\dots,\hat{R}^\ell$ approximate the reduced boundary $\partial^*_{g^\pm} v$ (see Subsection 3.1) in the following sense:

$$\int_{\hat{R}^i} |\mathscr{X}_{\hat{R}^i}(x) - v \vee g^\pm(x)|\, dx \leqslant \delta \operatorname{vol}(\hat{R}^i),$$

where $\mathscr{X}_{\hat{R}^i} = 1_{\hat{R}^{i,+}} - 1_{\hat{R}^{i,-}}$ and the volume of $\hat{R}^i$ is $\operatorname{vol}(\hat{R}^i) = \delta h^d$. Furthermore,

$$\left| \sum_{\hat{B}^i \cap \mathscr{P}_d = \varnothing} \tau_\beta(\vec{n}_i)\, |\hat{B}^i| + \Delta_{\beta, |\eta|} \sum_{\hat{B}^i \subseteq \mathscr{P}_d} |\hat{B}^i| - \mathscr{W}^{\pm}_{\beta, |\eta|}(v) \right| \leqslant \delta,$$

As we have mentioned, Theorem 4.1 (see Fig. 2) is proved as in the standard (+ b.c.) case, see ref. 9.

4.4. The Upper Bound

Proposition 4.2. We fix $\beta \in \mathfrak{B}$ and $\eta < 0$. For all v in $\mathrm{BV}(\operatorname{int} \hat{\mathbb{D}}, \{\pm 1\})$ such that $\mathscr{W}_{\beta,\eta}(v)$ is finite, one can choose $\delta_0 = \delta_0(v)$, such that uniformly in $\delta < \delta_0$

$$\limsup_{N \to \infty} \frac{1}{N^{d-1}} \max_{K_0(\delta) \leqslant K \leqslant N^\nu} \log \mu^{\beta,\eta}_{N,+}\left(\|\mathscr{M}_{N,K} - m^* v\|_{\mathbb{L}_1} \leqslant \delta\right) \leqslant -\mathscr{W}_{\beta,\eta}(v) + o(\delta),$$

where the function $o(\cdot)$ depends only on β and v and vanishes as δ goes to 0.

We fix K large enough and $\delta > 0$. Using the approximation Theorem, there is $\delta_0 < \delta$ such that for any $\delta' < \delta_0$

$$\begin{aligned}
&\limsup_{N \to \infty} \frac{1}{N^{d-1}} \log \mu^{\beta,\eta}_{N,+}\left(\left\| \frac{1}{m^*} \mathscr{M}_{N,K} - v \right\|_{\mathbb{L}_1} \leqslant \delta' \right) \\
&\quad \leqslant \limsup_{N \to \infty} \frac{1}{N^{d-1}} \log \mu^{\beta,|\eta|}_{N,\pm}\left(\frac{\mathscr{M}_{N,K}}{m^*} \in \bigcap_{i=1}^{\ell} \mathscr{V}(\hat{R}^i, 2\delta \operatorname{vol} \hat{R}^i) \right) \\
&\quad = \limsup_{N \to \infty} \frac{1}{N^{d-1}} \log \mathbb{P}^{\beta,|\eta|}_{N,\pm}\left(\frac{\mathscr{M}_{N,K}}{m^*} \in \bigcap_{i=1}^{\ell} \mathscr{V}(\hat{R}^i, 2\delta \operatorname{vol} \hat{R}^i) \right),
\end{aligned}$$

where $\mathscr{V}(\hat{R}^i, \varepsilon)$ is the ε-neighborhood of $\mathscr{X}_{\hat{R}^i}$

$$\mathscr{V}(\hat{R}^i, \varepsilon) = \left\{ v' \in \mathbb{L}_1(\hat{\mathbb{D}}) \,\middle|\, \int_{\hat{R}^i} |v'(x) - \mathscr{X}_{\hat{R}^i}(x)|\, dx \leqslant \varepsilon \right\}.$$

According to Lemma 3.1, there exist K_0 and ζ such that

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \max_{K_0 \leqslant K \leqslant N^\nu} \log \mathbb{P}_{N,\pm}^{\beta,|\eta|}\left(\frac{\mathscr{M}_{N,K}}{m^*} \in \bigcap_{i=1}^{\ell} \mathscr{V}(\hat{R}^i, 2\delta \operatorname{vol} \hat{R}^i)\right)$$
$$\leqslant \limsup_{N\to\infty} \frac{1}{N^{d-1}} \max_{K_0 \leqslant K \leqslant N^\nu} \log \mathbb{P}_{N,\pm}^{\beta,|\eta|}\left(u_{N,K}^{\zeta} \in \bigcap_{i=1}^{\ell} \mathscr{V}(\hat{R}^i, 3\delta \operatorname{vol} \hat{R}^i)\right).$$

The first step to extract the surface tension factor is to localize in each R_N^i a mesoscopic interface (where R_N^i is the microscopic counterpart of $\hat{R}^i$). This is done by means of a surgical procedure: in each R_N^i, the $\mathbb{L}_1$-constraint $u_{N,K}^{\zeta} \in \mathscr{V}(\hat{R}^i, 3\delta \operatorname{vol}(\hat{R}^i))$ enables to find two sections in $R_N^{i,+}$ and in $R_N^{i,-}$ (the minimal sections) containing only a small portion of the mesoscopic interface. So that up to a small cost one can rearrange the configurations in these sections in order to identify clearly the location of the interface. At that point the microscopic structure of Pisztora's coarse graining becomes effective (see Remark 3.1) and the presence of a mesoscopic interface is enough to ensure that the sets $\partial^{\mathrm{top}} R_N^{i\prime}$ and $\partial^{\mathrm{bot}} R_N^{i\prime}$ are disconnected on the microscopic level, where $\hat{R}^{i\prime}$ is the parallelepiped included in $\hat{R}^i$ with basis $\hat{B}^i$ and height $\frac{\delta}{2} h$. Therefore, introducing the set

$$\mathscr{A} = \left\{\omega \in \mathfrak{I} : \exists \sigma \text{ such that } u_{N,K}^{\zeta}(\sigma,\omega) \in \bigcap_{i=1}^{\ell} \mathscr{V}(\hat{R}^i, 3\delta \operatorname{vol}(\hat{R}^i))\right\},$$

we obtain

$$\mathbb{P}_{N,\pm}^{\beta,|\eta|}(\mathscr{A}) = \Phi_{N,\pm}^{\beta,|\eta|}(\mathscr{A}) \leqslant e^{C\,\delta N^{d-1}}\, \Phi_{N,\pm}^{\beta,|\eta|}\left(\bigcap_i \{\partial^{\mathrm{top}} R_N^{i\prime} \nleftrightarrow \partial^{\mathrm{bot}} R_N^{i\prime}\}\right), \tag{4.10}$$

The constant C in the error term is proportional to the perimeter of v. It is important to note that the surgical procedure is insensitive to the boundary effects and applies equally well near the wall $\mathscr{P}_d$ or in the bulk.

Equation (4.10) can then be bounded by

$$\Phi_{N,\pm}^{\beta,|\eta|}\left(\bigcap_i \{\partial^{\mathrm{top}} R_N^{i\prime} \nleftrightarrow \partial^{\mathrm{bot}} R_N^{i\prime}\}\right) \leqslant \frac{\Phi_N^{\beta,|\eta|,\mathrm{w}}(\bigcap_i \{\partial^{\mathrm{top}} R_N^{i\prime} \nleftrightarrow \partial^{\mathrm{bot}} R_N^{i\prime}\})}{\Phi_N^{\beta,|\eta|,\mathrm{w}}(\mathfrak{I})}. \tag{4.11}$$

As in the proof of the lower bound, the denominator can be estimated by (2.7). In order to recover the surface tension in each box R_N^i, we fix the boundary conditions outside each box R_N^i. At that point the relaxed definition of surface tension is crucial and we apply Lemma 2.1 or 2.2 depending whether the box R_N^i lies on the wall or in the bulk. This provides the following upper bound

$$\Phi_N^{\beta, |\eta|, \mathrm{w}}(\mathscr{A}) \leqslant \exp\left\{-N^{d-1}\left(\sum_{\hat{B}^i \cap \mathscr{P}_d = \varnothing} \tau_\beta(\vec{n}_i)\,|\hat{B}^i| + \Delta_{\beta, |\eta|} \sum_{\hat{B}^i \subseteq \mathscr{P}_d} |\hat{B}^i|\right) + C\,\delta N^{d-1}\right\}. \tag{4.12}$$

Using the approximation Theorem 4.1, we, thereby, obtain:

$$\Phi_N^{\beta, |\eta|, \mathrm{w}}(\mathscr{A}) \leqslant \exp\{-N^{d-1}\mathscr{W}^{\pm}_{\beta, |\eta|}(v) + o(\delta)\, N^{d-1}\}. \tag{4.13}$$

In view of (4.8) and the expression (3.4) for the functional $\mathscr{W}^{\pm}_{\beta, |\eta|}$ the proof of Proposition 4.2 is completed. ▮

4.5. Proof of Theorems 1.1 and 1.2

The Theorems 1.1 and 1.2 follow from Propositions 3.1, 4.1, 4.2. We will focus on the derivation of Theorem 1.2 in the case of negative boundary field, the other cases can be deduced easily.

The first step is to derive the following lower bound for $m \in\,]\bar{m}(\beta, \eta), m^*_\beta[$

$$\liminf_{N \to \infty} \frac{1}{N^{d-1}} \log \mu^{\beta, \eta}_{\mathbb{D}_N, +}(\mathbf{M}_{\mathbb{D}_N} \leqslant m) \geqslant -w^\star_{\beta, \eta}(m), \tag{4.14}$$

where $w^\star_{\beta, \eta}(m)$ was introduced in (1.11).

Fix ε in $]0, m^*_\beta - \bar{m}(\beta, \eta)[$. For any $\delta' > 0$ small enough and α in $(0, 1/d)$, one has

$$\mu^{\beta, \eta}_{\mathbb{D}_N, +}(\mathbf{M}_{\mathbb{D}_N} \leqslant m) \geqslant \mu^{\beta, \eta}_{\mathbb{D}_N, +}(\|\mathscr{M}_{N, N^\alpha} - m^* \mathbb{1}_{\mathscr{K}_{\beta, \eta}(v(m-\varepsilon))}\|_{\mathbb{L}_1} \leqslant \delta').$$

By using Proposition 4.1 and letting δ' go to 0, we see that

$$\liminf_{N \to \infty} \frac{1}{N^{d-1}} \log \mu^{\beta, \eta}_{\mathbb{D}_N, +}(\mathbf{M}_{\mathbb{D}_N} \leqslant m) \geqslant -w^\star_{\beta, \eta}(m - \varepsilon).$$

As the minimizers $\mathscr{K}_{\beta,\eta}(v(m))$ are obtained by dilatation of the same unnormalized Winterbottom shape $\mathscr{K}_{\beta,\eta}$, it follows that

$$\lim_{\varepsilon\to 0} w^{\star}_{\beta,\eta}(m-\varepsilon)=w^{\star}_{\beta,\eta}(m).$$

This concludes the proof of (4.14).

In order to derive the upper bound, we define for a given $\delta>0$

$$F_\delta=\left\{u\in\mathbb{L}^1(\hat{\mathbb{D}})\,\middle|\,\inf_x \|u-\mathbb{1}_{x+\mathscr{K}_{\beta,\eta}(v(m))}\|_{\mathbb{L}_1}\geqslant\delta;\int_{\hat{\mathbb{D}}}u(y)\,\mathrm{d}y\leqslant\frac{m}{m^*}\right\}.$$

For a given $a>0$, $\delta'>0$,

$$\mu^{\beta,\eta}_{\mathbb{D}_N,+}\left(\frac{1}{m^*}\mathscr{M}_{N,K}\in F_\delta\right)\leqslant\mu^{\beta,\eta}_{\mathbb{D}_N,+}\left(\frac{1}{m^*}\mathscr{M}_{N,K}\in F_\delta\cap\mathscr{V}(\mathscr{C}_{g^\pm}(a),\delta')\right)$$
$$+\mu^{\beta,\eta}_{\mathbb{D}_N,+}\left(\frac{1}{m^*}\mathscr{M}_{N,K}\notin\mathscr{V}(\mathscr{C}_{g^\pm}(a),\delta')\right).$$

According to Proposition 3.1, there is K_0 depending on δ' such that uniformly in a

$$\limsup_{N\to\infty}\frac{1}{N^{d-1}}\max_{K_0\leqslant K\leqslant N^\nu}\log\mathbb{P}^{\beta,|\eta|}_{N,\pm}\left(\frac{1}{m^*}\mathscr{M}_{N,K}\notin\mathscr{V}(\mathscr{C}_{g^\pm}(a),\delta')\right)$$
$$\leqslant-\frac{C_\beta}{K_0^{d-1}}\,a+|\Delta_{\beta,\eta}|.$$

We fix $\varepsilon>0$ and a much larger than $\frac{K_0^{d-1}}{C_\beta}(w^*_{\beta,\eta}(m)+|\Delta_{\beta,\eta}|)$. The exponential tightness will enable us to consider only a finite number of macroscopic configurations:

Since $\mathscr{C}_{g_\pm}(a)\cap F_\delta$ is compact in $\mathbb{L}_1(\hat{\mathbb{D}})$, we, using the upper bound Proposition 4.2, can cover it (and hence also $\mathscr{V}(\mathscr{C}_{g^\pm}(a)\cap F_\delta,\delta')$ for δ' sufficiently small) with a finite net of neighborhoods $(\mathscr{V}(u_i,\varepsilon_i))_{i\leqslant\ell}$ and choose a finite scale $K_1(\varepsilon)$ such that for every $i\leqslant\ell$

$$\limsup_{N\to\infty}\frac{1}{N^{d-1}}\max_{K_1(\varepsilon)\leqslant K\leqslant N^\nu}\log\mu^{\beta,\eta}_{N,+}\left(\frac{1}{m^*}\mathscr{M}_{N,K}\in\mathscr{V}(u_i,\varepsilon_i)\right)\leqslant-\mathscr{W}_{\beta,\eta}(u_i)+\varepsilon.$$

By the stability property (3.5),

$$\inf_{u\in F_\delta}\mathscr{W}_{\beta,\eta}(u)>w^{\star}_{\beta,\eta}(m).$$

Therefore, by choosing ε small enough, we can find a mesoscopic scale K_2 and a positive constant $c = c(\delta) > 0$, such that

$$\limsup_{N \to \infty} \frac{1}{N^{d-1}} \max_{K_2 \leqslant K \leqslant N^{\nu}} \log \mu_{\mathbb{D}_N, +}^{\beta, \eta} \left(\frac{1}{m^*} \mathscr{M}_{N,K} \in F_\delta \right) \leqslant -w_{\beta,\eta}^{\star}(m) - c(\delta).$$

The above inequality combined with the lower bound (4.14) leads to the result. ∎

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